# An integral arising from the chiral $sl(n)$ Potts model


Anthony J. Guttmann[1]

*ARC Centre of Excellence for*
*Mathematics and Statistics of Complex Systems,*
*Department of Mathematics and Statistics,*
*The University of Melbourne, Victoria 3010, Australia*

Mathew D Rogers[2]

*Department of Mathematics and Statistics,*
*Université de Montréal, Montréal, Québec, H3C 3J7, Canada*



**Abstract**

We show that the integral

$$J(t) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi dxdydz \log(t - \cos x - \cos y - \cos z + \cos x \cos y \cos z), \qquad (1)$$

can be expressed in terms of ${}_5F_4$ hypergeometric functions. The integral arises in the solution by Baxter and Bazhanov of the free-energy of the $sl(n)$ Potts model, which includes the term $J(2)$. Our result immediately gives the logarithmic Mahler measure of the Laurent polynomial

$$k - \left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right) - \left(z + \frac{1}{z}\right) + \frac{1}{4}\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right)$$

in terms of the same hypergeometric functions.


## 1 Calculation of the integral

There exists an extensive literature on solvable two-dimensional lattice models, based on the concept of commuting transfer matrices and the Yang-Baxter relation. Far fewer solutions are available for three-dimensional models. The first such model was obtained by Zamolodchikov [17, 18] in 1980-81. He introduced a so-called tetrahedron relation as the appropriate generalization of the Yang-Baxter equation. In 1983 Baxter solved this model using only commutativity, symmetry and a factorizability property of the transfer matrix. A decade later Bazhanov and Baxter [2] studied the $sl(n)$-chiral Potts model [3], which can be considered a multi-state generalization of the Zamolodchikov model, and showed that for this model it was also possible to bypass the tetrahedron equations and


[1]email: `tonyg@ms.unimelb.edu.au`
[2]email: `mathewrogers@gmail.com`

solve the model by using symmetry properties to prove the commutativity of the row-to-row transfer matrices. Aspects of the solution were later discussed by Bazhanov [4], who showed that the solution could be expressed in terms of a free boson or Gaussian model on the three-dimensional cubic lattice. The partition function was expressed in terms of a fractional power of the determinant of a cyclic square matrix. In the thermodynamic limit the free energy was given as the sum of two terms, one of which was the integral $J(2)$, where

$$J(t) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi dk_1 dk_2 dk_3 \log(t - c_1 - c_2 - c_3 + c_1c_2c_3), \tag{2}$$

where $c_i$ denotes $\cos(k_i)$. Baxter and Bazhanov showed that

$$J(2) = \frac{8}{\pi}L_{-4}(2) - 3\log 2, \tag{3}$$

where $L_{-4}(2)$ is Catalan's constant, $L_k(s) := \sum_{n=1}^\infty \left(\frac{k}{n}\right) n^{-s}$ denotes Dirichlet's $L$-series, and $\left(\frac{k}{n}\right)$ is the Jacobi symbol.

Here we show that the integral can be expressed in terms of ${}_5F_4$ hypergeometric functions for $|t|$ sufficiently large, and $t \geq 2.0802\ldots$ on the real axis. Our starting point is the work of Delves, Joyce and Zucker [10] who considered the related, Green function integral

$$G(t) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dk_1 dk_2 dk_3}{t - c_1 - c_2 - c_3 + c_1c_2c_3}. \tag{4}$$

If $t$ lies in the complex plane cut along the real axis from $-2$ to $2$, then they proved that

$$G(t) = \frac{1}{t}\left(1-\frac{4}{t^2}\right)^{-1/4}\left[{}_2F_1\left({\frac{1}{8},\frac{5}{8} \atop 1};\frac{4}{t^2}\right)\right]^2. \tag{5}$$

There are at least two methods for integrating this formula. The more difficult approach is to use (5) to derive a modular expansion for $J(t)$, which can then be compared to known modular expansions for ${}_5F_4$ functions. This approach was used in [14] to study three-variable Mahler measures. We leave those calculations for a future paper [15], since the following theorem can be proved via standard hypergeometric transformations.

**Theorem 1.** *Suppose that $|\alpha|$ is sufficiently small but non-zero. Then*

$$\begin{aligned} &J\left(\sqrt{4\alpha(1-\alpha)} + \frac{1}{\sqrt{4\alpha(1-\alpha)}}\right) \\ &\quad= -\frac{1}{2}\log\left(4\alpha(1-\alpha)^{19}(1+\alpha)^{12}\right) - \frac{11}{4}\alpha(1-\alpha){}_5F_4\left({\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2,2};4\alpha(1-\alpha)\right) \\ &\qquad - \frac{7\alpha}{4(1-\alpha)^2}{}_5F_4\left({\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2,2};-\frac{4\alpha}{(1-\alpha)^2}\right) + \frac{9\alpha(1-\alpha)^2}{4(1+\alpha)^4}{}_5F_4\left({\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1 \atop 2,2,2,2};\frac{16\alpha(1-\alpha)^2}{(1+\alpha)^4}\right). \end{aligned} \tag{6}$$

*Proof.* We show that the derivatives of both sides of the equation agree. To simplify the calculations, briefly assume that $\alpha$ is a small positive real number. Notice that for $s \in (0,1)$:

$$\frac{d}{dz}\left[z\;{}_5F_4\left({2-s,\frac{3}{2},1+s,1,1 \atop 2,2,2,2};z\right)\right] = \frac{2}{s(1-s)z}\left[{}_3F_2\left({1-s,\frac{1}{2},s \atop 1,1};z\right)-1\right]. \tag{7}$$

Let us set

$$t := \sqrt{4\alpha(1-\alpha)} + \frac{1}{\sqrt{4\alpha(1-\alpha)}}. \tag{8}$$

Differentiate (6) with respect to $\alpha$, and then apply (4), (5), and (7), to obtain the presumed equality

$$\begin{aligned}\frac{1}{t}\left(1-\frac{4}{t^2}\right)^{-1/4}&\left[{}_2F_1\left({\frac{1}{8},\frac{5}{8} \atop 1};\frac{4}{t^2}\right)\right]^2 \ \frac{2(2\alpha-1)^3}{(4\alpha(1-\alpha))^{3/2}}\\ &= -\frac{11(1-2\alpha)}{2\alpha(1-\alpha)}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right) + \frac{7(1+\alpha)}{2\alpha(1-\alpha)}\,{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};-\frac{4\alpha}{(1-\alpha)^2}\right)\\ &\quad+\frac{3\left(1-6\alpha+\alpha^2\right)}{2\alpha\left(1-\alpha^2\right)}\,{}_3F_2\left({\frac{1}{4},\frac{1}{2},\frac{3}{4} \atop 1,1};\frac{16\alpha(1-\alpha)^2}{(1+\alpha)^4}\right).\end{aligned} \tag{9}$$

We can re-express the left-hand side of the identity using equation (16) on p. 112 of Erdélyi et al. [12], giving

$$\left(1-\frac{4}{t^2}\right)^{-1/4}\left[{}_2F_1\left({\frac{1}{8},\frac{5}{8} \atop 1};\frac{4}{t^2}\right)\right]^2 = \xi\left[{}_2F_1\left({\frac{1}{4},\frac{3}{4} \atop 1};\frac{1-\xi}{2}\right)\right]^2,$$

where $\xi = \left(1-\frac{4}{t^2}\right)^{-1/2}$. Substituting for $t$ yields

$$\begin{aligned}\left(1-\frac{4}{t^2}\right)^{-1/4}\left[{}_2F_1\left({\frac{1}{8},\frac{5}{8} \atop 1};\frac{4}{t^2}\right)\right]^2 &= \frac{1+4\alpha-4\alpha^2}{(1-2\alpha)^2}\left[{}_2F_1\left({\frac{1}{4},\frac{3}{4} \atop 1};-\frac{4\alpha(1-\alpha)}{(1-2\alpha)^2}\right)\right]^2\\ &= \frac{1+4\alpha-4\alpha^2}{(1-\alpha)(1-2\alpha)}\left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1},\frac{\alpha}{\alpha-1}\right)\right]^2\\ &= \frac{1+4\alpha-4\alpha^2}{(1-2\alpha)}\left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1},\alpha\right)\right]^2,\end{aligned} \tag{10}$$

where the second and third steps follow from [5, pg. 95] and [5, pg. 38] The right-hand side of (9) simplifies via Clausen's identity, and the same quadratic transformations:

$${}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right) = \left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha\right)\right]^2, \tag{11}$$

$${}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,1};-\frac{4\alpha}{(1-\alpha)^2}\right) = \left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\frac{\alpha}{\alpha-1}\right)\right]^2 = (1-\alpha)\left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha\right)\right]^2 \tag{12}$$

$${}_3F_2\left({\frac{1}{4},\frac{1}{2},\frac{3}{4} \atop 1,1};\frac{16\alpha(1-\alpha)^2}{(1+\alpha)^4}\right) = \left[{}_2F_1\left({\frac{1}{4},\frac{3}{4} \atop 1};\frac{4\alpha}{(1+\alpha)^2}\right)\right]^2 = (1+\alpha)\left[{}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};\alpha\right)\right]^2 \tag{13}$$

If we substitute (8), (10), (11), (12), and (13) into (9), we see that (9) holds whenever $\alpha$ is a sufficiently small real number. This implies that (6) holds up to a constant of integration. The constant of integration is easily seen to equal zero, because both sides of (6) approach $-\frac{1}{2}\log\alpha + 0$ when $\alpha$ tends to zero. Finally, if we add $\frac{1}{2}\log\alpha$ to either side of the identity, then both sides are analytic in a neighborhood of $\alpha = 0$, so (6) holds for $|\alpha|$ sufficiently small but non-zero. □

The formula for $J(t)$ holds on the positive real axis for $0<\alpha\leq(\sqrt{2}-1)^2\approx .1715...$, which implies $t\geq 2.0802\ldots$. The identity fails when $\alpha>(\sqrt{2}-1)^2$, because the argument of the third hypergeometric function crosses a branch cut which lies on $[1,\infty)$. In general, we can analytically continue (6) along a ray starting from $\alpha=0$, and ending at a point where one of the functions ceases to be analytic. Despite the fact that equation (6) can not be used to calculate $J(2)$, it is still possible to reprove the formula of Baxter and Bazhanov via a closely related modular expansion [15]:

$$\begin{aligned} J\left(u\left(e^{-2\pi v}\right)\right)=&-3\log 2+\frac{15v}{\pi^3}\sum_{(n,k)\neq(0,0)}\frac{3n^2-(2v)^2k^2}{\left(n^2+(2v)^2k^2\right)^3}\\ &+\frac{48v}{\pi^3}\sum_{n,k}\frac{3(2n+1)^2-(2v)^2(2k+1)^2}{\left((2n+1)^2+(2v)^2(2k+1)^2\right)^3}, \end{aligned} \tag{14}$$

where

$$u(q)=\left(\sqrt[4]{2}\frac{\eta(q)\eta\left(q^4\right)}{\eta^2\left(q^2\right)}\right)^{12}+\left(\sqrt[4]{2}\frac{\eta(q)\eta\left(q^4\right)}{\eta^2\left(q^2\right)}\right)^{-12}, \tag{15}$$

and $\eta(q)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. Equation (14) can be derived by combining equation (6) with (2.12) and (2.14) in [14]. It is possible to show that (14) holds for $v\geq\frac{1}{2}$, and that the left-hand side of the identity equals $J(2)$ when $v=\frac{1}{2}$. The right-hand side reduces to two-dimensional lattice sums, which can ultimately be evaluated by appealing to results of Glasser and Zucker [11]. Bertin, Rodriguez-Villegas, and Samart have all used a similar modular approach to prove Mahler measure formulas [7], [13], [16].

Equation (6) also implies an identity between Mahler measures. The Mahler measure of an $n$-variable Laurent polynomial, $P(x_1,\ldots,x_n)$, is defined by

$$\mathrm{m}(P):=\int_0^1\cdots\int_0^1\log\left|P\left(e^{2\pi i t_1},\ldots,e^{2\pi i t_n}\right)\right|dt_1\ldots dt_n.$$

The second author related both ${}_5F_4$ functions to Mahler measures in [14]. After some simplification, equation (6) implies

$$\begin{aligned} &\mathrm{m}\left(8\sqrt{4\alpha(1-\alpha)}+\frac{8}{\sqrt{4\alpha(1-\alpha)}}-4\left(x+x^{-1}+y+y^{-1}+z+z^{-1}\right)+\left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right)\\ &\quad=11\mathrm{m}\left(\frac{4}{\sqrt{\alpha(1-\alpha)}}+\left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right)\\ &\qquad-7\mathrm{m}\left(\frac{4i(1-\alpha)}{\sqrt{\alpha}}+\left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right)\\ &\qquad-6\mathrm{m}\left(x^4+y^4+z^4+1+\frac{2(1+\alpha)}{\sqrt[4]{\alpha(1-\alpha)^2}}xyz\right), \end{aligned}$$

and this identity also holds on the real axis for $\alpha\in(0,(\sqrt{2}-1)^2]$.

## 2 Conclusion and special values of $J(t)$

We conclude by noting that formula (3) of Baxter and Bazhanov is not an isolated result, and that there are many additional explicit formulas for values of $J(t)$. Most of these formulas involve $L$-functions of eta products. These are not elementary constants, but they often carry deep number-theoretic significance, so in some sense they can still be regarded as fundamental constants. For more details on explicit Mahler measure formulas, we refer to the work of Boyd [8]. We conclude with a single example of such a formula. If we have a function

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau},$$

then the $L$-series associated with $f$ is defined by

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Thus if we take $\eta(\tau)$ to be the usual Dedekind eta function, it is possible to prove

$$J\left(\frac{5}{2}\right) = \frac{24\sqrt{3}}{\pi^3} L\left(\eta^3(2\tau)\eta^3(6\tau), 3\right) + \frac{15\sqrt{3}}{4\pi} L_{-3}(2) - 3\log 2. \tag{16}$$

We have also obtained more complicated identities for $J(14)$, $J(322)$, and $J(t)$ for various irrational algebraic values of $t$ [15].

It is also noteworthy that the two hypergeometric functions that appear in Theorem 1 are precisely those that appear in our solution [9] of the spanning tree constant for the simple-cubic lattice.

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