

ON THE MAHLER MEASURE OF $1 + X + 1/X + Y + 1/Y$

MATHEW ROGERS AND WADIM ZUDILIN

ABSTRACT. We prove a conjectured formula relating the Mahler measure of the Laurent polynomial $1 + X + X^{-1} + Y + Y^{-1}$ to the L -series of a conductor 15 elliptic curve.

1. INTRODUCTION

The purpose of this paper is to prove a conjectured identity relating the Mahler measure of a two-variable Laurent polynomial, to the L -series of a conductor 15 elliptic curve [9]:

$$(1) \quad m\left(1 + X + \frac{1}{X} + Y + \frac{1}{Y}\right) = \frac{15}{4\pi^2}L(E_{15}, 2).$$

Recall that the (logarithmic) Mahler measure of a polynomial $P(X_1, \dots, X_n) \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is the arithmetic mean of $\log |P|$ on the torus $\mathbb{T}^n = \{(X_1, \dots, X_n) \in \mathbb{C}^n : |X_1| = \dots = |X_n| = 1\}$,

$$(2) \quad m(P) := \int \cdots \int_{[0,1]^n} \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n.$$

The study of multi-variable Mahler measures originated in the work of Smyth, who proved relations with Dirichlet L -values and special values of the Riemann zeta function [22]. Formula (1) is the first known relation between a Mahler measure and the L -series of an elliptic curve. The original formulation is due to Deninger, who proved that the identity follows, up to an unknown rational factor, from the Beilinson conjectures [9]. Boyd subsequently calculated the rational factors, and also found [5] that similar identities were numerically true for the polynomial family $k + X + X^{-1} + Y + Y^{-1}$ whenever $k \in \mathbb{Z}$. Bertin [2] and Rodriguez-Villegas [18] have also investigated Mahler measures of elliptic curves.

The authors recently proved Boyd's conjectures for non-CM elliptic curves of conductors 20 and 24 [21]. The basic idea was to manipulate the cusp forms associated with the elliptic curves, in order to obtain elementary integrals for the L -values. In

Date: March 29, 2010.

2000 Mathematics Subject Classification. Primary 33C20; Secondary 11F03, 14H52, 19F27, 33C75, 33E05.

Key words and phrases. Mahler measure, L -value of elliptic curve, modular equation, hypergeometric series, lattice sum, elliptic dilogarithm.

The first author is supported by National Science Foundation award DMS-0803107. The second author is supported by Australian Research Council grant DP110104419.

the conductor 20 case, it was shown that

$$(3) \quad L(E_{20}, 2) = -\frac{\pi}{20} \int_0^1 \frac{(1-6t) \log(1+4t)}{\sqrt{t(1-t)(1+4t^2)}} dt.$$

The integrals were then related to Mahler measures through an intricate analysis of hypergeometric functions. Formula (3) can be reduced to a Mahler measure by setting $k = 4$ in an identity valid for $k \in [2, 8]$:

$$\frac{1}{2\pi} \int_0^1 \frac{(2-k+3kt) \log(1+kt)}{\sqrt{t(1-t)(4+(4-k)kt+k^2t^2)}} dt = m((1+X)(1+Y)(X+Y) - kXY).$$

As might be expected, these sorts of integrals are difficult to analyze. It required a great deal of trial and error to introduce the parameter k into (3). Even with the correct definition, it was very difficult to relate the integral to a Mahler measure.

We will use elementary techniques to prove formula (1). Our method relies upon integrating Ramanujan's modular equations, and is applicable to many different elliptic curves. Prior to our work, the only method for attacking Boyd's conjectures centered around Beilinson's theorem. Brunault and Mellit used Beilinson's theorem to prove Boyd's conjectures for conductor 11 and 14 elliptic curves [8], [16]. We expect to present elementary proofs of their results in a future paper. It is important to mention the fact that Zagier and Kontsevich predicted the existence of formulas like (3), as a consequence of Beilinson's theorem [12, §3.4]. While our current method is independent of such K -theoretic considerations, it seems that the two approaches yield overlapping results. We believe that this current work, and our previous paper [21], are the first instances where elementary formulas such as (3) have been explicitly stated.

We will introduce two new ideas in this paper. The first is that it is possible to express *many* different L -values in terms of a single function $H(x)$, the function introduced in [21]. If we consider the eta function with respect to q ,

$$\eta(q) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

and define the signature 3 theta functions [1, Chap. 33], [3]

$$(4) \quad a(q) := \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2}, \quad b(q) := \sum_{m,n \in \mathbb{Z}} e^{2\pi i(m-n)/3} q^{m^2+mn+n^2} = \frac{\eta^3(q)}{\eta(q^3)},$$

and $c(q) := \sum_{m,n \in \mathbb{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} = 3 \frac{\eta^3(q^3)}{\eta(q)},$

then $H(x)$ is given by

$$(5) \quad H(x) := \int_0^1 \frac{\eta^3(q^3)}{\eta(q)} \frac{\eta^3(q^x)}{\eta(q^{3x})} \log q \frac{dq}{q} = \frac{1}{3} \int_0^1 b(q^x) c(q) \log q \frac{dq}{q}.$$

If we recall that the conductor 27 elliptic curve is associated to $\eta^2(q^3)\eta^2(q^9)$ [15], then it is simple to see that

$$-9L(E_{27}, 2) = H(1).$$

It is much less obvious that the L -series of a conductor 15 elliptic curve also reduces to values of $H(x)$. We will use a telescoping modular equation to prove that

$$(6) \quad -45L(E_{15}, 2) = \frac{1}{5}H\left(\frac{1}{15}\right) + 5H\left(\frac{5}{3}\right) + \frac{4\pi^2}{3}\log 3.$$

We have discovered that at least 9 different L -values can be related to $H(x)$; those formulas are presented in the next section. The definition of $H(x)$ was initially guessed after examining the complex-multiplication, conductor 27, example. There are several additional functions which possess properties analogous to $H(x)$. Those functions can be used to prove many additional relations between Mahler measures and L -series of elliptic curves, and will be examined in forthcoming papers.

The second idea we will require, is that certain linear combinations of $H(x)$ can be reduced to elementary integrals. These identities go well beyond the scope of our previous analysis in [21]. For any $x \in \mathbb{Q} \cap (0, \infty)$, there exists a polynomial relation between u and v , such that

$$(7) \quad \begin{aligned} & xH\left(\frac{x}{3}\right) + \frac{1}{x}H\left(\frac{1}{3x}\right) + 2H\left(\frac{1}{3}\right) \\ & = 4\pi \int_{v \in [0,1]} \log\left(\frac{1-R+v}{u}\right) d \arctan\left(\frac{\sqrt{3}(1+R)}{1-R-2v}\right), \end{aligned}$$

where $R^3 - 3vR - v^3 = 1 - u^3$. The right-hand side of (7) is essentially a function of a polynomial. While the value of x dictates the choice of polynomial, it should be possible to evaluate the integral for any sufficiently simple algebraic relation between u and v . When $x = 2$ the corresponding relation is $u + v - 1 = 0$, and when $x = 5$ the relation is given by $(u + v - 1)^2 - 9uv = 0$. We will use formulas (6) and (7) to prove the conductor 15 conjecture.

2. TELESCOPING MODULAR EQUATIONS AND NUMERICAL CONJECTURES

In this section we will prove formulas relating L -values to $H(x)$ defined in (5). We will also present some unproven formulas, which hold to high numerical precision. A number of similar formulas were proved in [21]. The basic idea in the previous paper, was to decompose cusp forms into signature 3 theta functions. For instance, integrating the formula

$$(8) \quad 3\eta^4(q^6) = b(q^4)c(q^3) - b(q)c(q^{12}),$$

leads to a linear relation between $L(E_{36}, 2)$, $H(4/3)$ and $H(1/12)$ [21]. Recall that the L -series of a conductor 36 elliptic curve equals the Mellin transform of $\eta^4(q^6)$ [15]. We can state this result in the form $L(E_{36}, 2) = F(1, 1)$, where the quadruple

sum

(9)

$$F(B, C) := (B + 1)^2(C + 1)^2 \times \sum_{\substack{n_i=-\infty \\ i=1,2,3,4}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left((6n_1 + 1)^2 + B(6n_2 + 1)^2 + C(6n_3 + 1)^2 + BC(6n_4 + 1)^2\right)^2}$$

can be identified as the L -series $L(f, 2)$ of the cusp form

$$f(q) = \eta(q^A)\eta(q^{AB})\eta(q^{AC})\eta(q^{ABC}), \quad A = \frac{24}{(B + 1)(C + 1)},$$

whenever A is an integer.

Many of the new formulas in this section, follow from integrating ‘telescoping’ modular equations. The key is to search for identities which are similar to (8), but which involve additional terms. When the telescoping terms are integrated, their modularity properties can be used to evaluate them explicitly. The following formula sets the grounds of this idea.

Lemma 1. *For $r > 0$ and $j > 0$,*

$$(10) \quad \int_0^1 (r^2 c(q^r) c(q^{rj}) - c(q) c(q^j)) \log q \frac{dq}{q} = \frac{4\pi^2}{3j} \log r.$$

We will illustrate the utility of this approach with an example. Consider the following modular equation:

$$(11) \quad 0 = a(q)a(q^2) - b(q)b(q^2) - c(q)c(q^2).$$

Formula (11) is a q -version of the second degree modular equation in Ramanujan’s theory of signature 3 [4, Theorem 2.6]. Eliminating $a(q)$ and $a(q^2)$ with a standard relation, $a(q) = b(q) + 3c(q^3)$, brings the equation to

$$3b(q)c(q^6) + 3b(q^2)c(q^3) = -9c(q^3)c(q^6) + c(q)c(q^2).$$

Now multiply both sides by $(2 \log q)/q$, and integrate for $q \in (0, 1)$. The left-hand side immediately reduces to values of $H(x)$. Applying Lemma 1 with $r = 3$ and $j = 2$, yields

$$(12) \quad \frac{1}{2}H\left(\frac{1}{6}\right) + 2H\left(\frac{2}{3}\right) = -\frac{4\pi^2}{3} \log 3.$$

Thus we have obtained a \mathbb{Q} -linear dependency between $H(1/6)$, $H(2/3)$, and $\pi^2 \log 3$. There are at least three variants of (11) which lead to formulas for L -functions of elliptic curves.

Proposition 1. *The following relations are either proved or hold numerically:*

$$(13) \quad \frac{4\pi^2}{3} \log 3 = -\frac{1}{x}H\left(\frac{1}{3x}\right) + 3xH\left(\frac{x}{3}\right) - 3xH(x) + \frac{1}{x}H\left(\frac{1}{9x}\right),$$

$$(14) \quad \pi\sqrt{3}L(\chi_{-3}, 2) = -H\left(\frac{1}{3}\right),$$

$$\begin{aligned}
(15) \quad 12L(E_{14}, 2) &= 12F(2, 7) = -\frac{1}{14^2}H\left(\frac{1}{42}\right) - H\left(\frac{14}{3}\right) \\
&\quad + \frac{1}{7^2}H\left(\frac{2}{21}\right) + \frac{1}{2^2}H\left(\frac{7}{6}\right), \\
(16) \quad 9L(E_{15}, 2) &= 9F(3, 5) = -\frac{1}{5^2}H\left(\frac{1}{15}\right) - H\left(\frac{5}{3}\right) - \frac{4\pi^2}{15}\log 3, \\
(17) \quad 24L(E_{20}, 2) &= 24F(1, 5) \stackrel{?}{=} -\frac{2}{5^2}H\left(\frac{1}{15}\right) + 2H\left(\frac{5}{3}\right) + \frac{4}{5^2}H\left(\frac{4}{15}\right) \\
&\quad - 4H\left(\frac{20}{3}\right) + \frac{3}{5}H\left(\frac{1}{3}\right), \\
(18) \quad 9L(E_{24}, 2) &= 9F(2, 3) = -\frac{1}{8^2}H\left(\frac{1}{24}\right) - H\left(\frac{8}{3}\right) - \frac{\pi^2}{6}\log 3, \\
(19) \quad 9L(E_{27}, 2) &= 9F(1, 3) = -H(1), \\
(20) \quad 9L(E_{36}, 2) &= 9F(1, 1) = -H\left(\frac{4}{3}\right) + \frac{1}{4^2}H\left(\frac{1}{12}\right) \\
(21) \quad &\stackrel{?}{=} 2H\left(\frac{1}{3}\right) - 2H\left(\frac{4}{3}\right), \\
(22) \quad 6L(E_{33}, 2) + 4L(E_{11}, 2) &= 12F(1, 11) + \frac{9}{2}F(3, 11) \\
&= -\frac{1}{11^2}H\left(\frac{1}{33}\right) - H\left(\frac{11}{3}\right) - \frac{4\pi^2}{33}\log 3, \\
(23) \quad \frac{27}{16}F(3, 7) &= \frac{8}{7}H(1) - H(7) - \frac{1}{49}H\left(\frac{1}{7}\right), \\
(24) \quad \frac{27}{49}F(6, 7) &= \frac{1}{49}H\left(\frac{2}{7}\right) + H(14) - \frac{8}{7}H(2), \\
(25) \quad \frac{27}{25}F\left(\frac{3}{2}, 7\right) &= \frac{2}{7}H\left(\frac{1}{2}\right) - \frac{1}{4}H\left(\frac{7}{2}\right) - \frac{1}{14^2}H\left(\frac{1}{14}\right),
\end{aligned}$$

where χ_{-3} in (14) denotes the non-principal character mod 3.

Proof. We will begin by proving formulas (16), (18) and (22). The proofs follow in exactly the same manner as the proof of (12), using the telescoping identity of Lemma 1. The relevant modular equations are due to Ramanujan:

$$\begin{aligned}
a(q)a(q^5) - b(q)b(q^5) - c(q)c(q^5) &= 9\eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}), \\
a(q)a(q^8) - b(q)b(q^8) - c(q)c(q^8) &= 9\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}), \\
a(q)a(q^{11}) - b(q)b(q^{11}) - c(q)c(q^{11}) &= 9\eta^2(q)\eta^2(q^{11}) + 27\eta^2(q^3)\eta^2(q^{33}) \\
&\quad + 18\eta(q)\eta(q^3)\eta(q^{11})\eta(q^{33}).
\end{aligned}$$

The first modular equation is equivalent to [1, pg. 125, Entry 7.20]. The equivalence follows from using product expansions $b(q) = \eta^3(q)/\eta(q^3)$, $c(q) = 3\eta^3(q^3)/\eta(q)$, and

the cubic relation $a(q) = (b^3(q) + c^3(q))^{1/3}$. The second result follows from [1, pg. 129, Entry 7.40], and the third follows from [1, pg. 127, Entry 7.30]. The identification of $L(E_{33}, 2)$ in terms of $F(1, 11)$ and $F(3, 11)$, follows from integrating the associated cusp form [17].

In order to prove (14), we can use [11, pg. 217, Entry 14.6], to obtain

$$\frac{1}{3}b(q)c(q^3) = \sum_{n,k=1}^{\infty} k\chi_{-3}(nk)q^{nk}.$$

Multiplying by $(\log q)/q$ and integrating for $q \in (0, 1)$ on either side, we have

$$\frac{1}{3^2}H\left(\frac{1}{3}\right) = -L(\chi_{-3}, 1)L(\chi_{-3}, 2) = -\frac{\pi}{3\sqrt{3}}L(\chi_{-3}, 2).$$

The proof of (13) follows from integrating the following identity:

$$(b(q^{1/9}) - b(q^{1/3}))c(q^x) + 3(b(q^{x/3}) - b(q^x))c(q) = 9c(q^{3x})c(q) - c(q^x)c(q^{1/3}),$$

and then applying (10) to the right-hand side. This identity can be verified by eliminating $b(q)$ with

$$b(q^{1/3}) - b(q) = 3c(q^3) - c(q).$$

The simpler identity between b and c is a consequence of [1, pg. 93, Entry 2.8] and [1, pg. 94, Entry 2.9].

Finally, identities (19), (20), (23) (24), and (25) are examined in [21, Lemma 1]. \square

Proof of Lemma 1. Let us denote the left-hand side of (10) by $I_{j,r}$. Notice that

$$I_{j,r} = \lim_{\delta \rightarrow 1} \left(\int_0^\delta r^2 c(q^r) c(q^{rj}) \log q \frac{dq}{q} - \int_0^\delta c(q) c(q^j) \log q \frac{dq}{q} \right).$$

The rearrangement is justified because $c(q) = O(q^{1/3})$ as $q \rightarrow 0^+$. Performing a u -substitution brings the difference to

$$I_{j,r} = \lim_{\delta \rightarrow 1} \int_\delta^{\delta^r} c(q) c(q^j) \log q \frac{dq}{q}.$$

It is known that $b(q)$ and $c(q)$ are linked by the modularity relation

$$c(q) = -\frac{2\pi}{\sqrt{3} \log q} b(e^{(4\pi^2)/(3 \log q)}).$$

When $q \rightarrow 1^-$ it is easy to see that $e^{(4\pi^2)/(3 \log q)} \rightarrow 0$. Since $b(0) = 1$, we estimate

$$c(q) = -\frac{2\pi}{\sqrt{3} \log q} (1 + O(1 - q)) \quad \text{as } q \rightarrow 1^-.$$

Substituting for $c(q)$ and $c(q^j)$ reduces the integral to

$$\begin{aligned} I_{j,r} &= \lim_{\delta \rightarrow 1} \frac{4\pi^2}{3j} \int_{\delta}^{\delta^r} \frac{1}{\log q} (1 + O(1-q)) \frac{dq}{q} \\ &= \frac{4\pi^2}{3j} \log r + \lim_{\delta \rightarrow 1} \int_{\delta}^{\delta^r} O\left(\frac{1-q}{q \log q}\right) dq \\ &= \frac{4\pi^2}{3j} \log r. \end{aligned}$$

The error term is the tail of the convergent integral

$$\int_{1/2}^1 \frac{1-q}{q \log q} dq,$$

and vanishes as $\delta \rightarrow 1$. This concludes the proof of (10). \square

To finish this discussion, we will emphasize the fact that formula (16) is the real prize of Proposition 1. That formula provides one of the keys to solving the conductor 15 conjecture. Formula (15) also seems promising, however we basically ignored the identity in our analysis, because Mellit has already proved Boyd's conjectures for conductor 14 curves [16]. The proof of (15) is also quite difficult, and as a result we have chosen not to present it here. This brings us to the numerical identities. We were disappointed that we could not isolate the conductor 11 case, since that L -value appears frequently in Boyd's tables. Equation (17) is the most interesting formula that we were able to conjecture, since it involves the conductor 20 elliptic curve. We discovered (17) with the PSLQ algorithm, and were subsequently unable to prove it. The problem is that there is no obvious way to relate integrals of $\eta^2(q^2)\eta^2(q^{10})$ to values of $H(x)$. We performed an extensive, but ultimately futile, search of Ramanujan's formulas [1] and Somos's identities [23]. Eventually we chose to bypass the problem, because Boyd's conductor 20 conjectures are already proved [21].

It seems likely that a more extensive search will turn up many additional results. Our primary goal was to find formulas for the lattice sums $F(B, C)$ defined in (9). Since the vast majority of elliptic curves have a value of $L(E, 2)$ which is (presumably) linearly independent from $F(B, C)$ over \mathbb{Q} , our search would not have addressed those cases. There are also many additional \mathbb{Q} -linear dependencies between values of $H(x)$. The majority of these formulas are probably insignificant, and we expect that most of them can be proved with a telescoping recipe. For instance, we calculated

$$\begin{aligned} 5H(1) &\stackrel{?}{=} 4H(4) + \frac{1}{4}H\left(\frac{1}{4}\right), \\ 3H(2) &\stackrel{?}{=} 4H\left(\frac{2}{3}\right) + \frac{1}{4}H\left(\frac{1}{18}\right). \end{aligned}$$

The only continuous identity that we discovered was (13). Perhaps it is noteworthy that this functional relation can be combined with our other results, to either prove or conjecture explicit formulas for $H(1)$, $H(1/3)$, $H(2/3)$, $H(1/6)$, and $H(1/9)$.

3. A NEW INTEGRAL FOR $H(x)$

One of our main theorems in [21], is that it is always possible to express $H(x)$ as an integral of elementary functions. Suppose that $x > 0$, and assume that β has degree x over α in the theory of signature 3. Then it was proved that

$$(26) \quad xH\left(\frac{x}{3}\right) = \frac{2\pi}{\sqrt{3}} \int_0^1 \frac{(1-\alpha)^{1/3}(1-(1-\alpha)^{1/3})}{\alpha(1-\alpha)} \log \frac{1-(1-\beta)^{1/3}}{\beta^{1/3}} d\alpha.$$

We say that β has degree x over α in signature 3, if α and β can be parameterized by

$$\alpha = \frac{c^3(q)}{a^3(q)}, \quad \beta = \frac{c^3(q^x)}{a^3(q^x)},$$

where $a(q)$ and $c(q)$ are the signature 3 theta functions (4). The existence of signature 3 modular equations is a consequence of the classical theory of modular forms. If $q = e^{2\pi i\tau}$, then α and β are algebraic functions of $j(\tau)$ and $j(x\tau)$, and therefore satisfy an algebraic relation whenever $x \in \mathbb{Q} \cap (0, \infty)$.

Proposition 2. *Suppose that $x > 0$. Then we have*

$$(27) \quad xH\left(\frac{x}{3}\right) + \frac{1}{x}H\left(\frac{1}{3x}\right) + 2H\left(\frac{1}{3}\right) \\ = 4\pi \int_{v \in [0,1]} \log\left(\frac{1-R+v}{u}\right) d \arctan\left(\frac{\sqrt{3}(1+R)}{1-R-2v}\right),$$

where $R^3 - 3vR - v^3 = 1 - u^3$. There is another algebraic relation between u and v whenever $x \in \mathbb{Q} \cap (0, \infty)$. This relation is induced by the parameterizations $u = (\alpha\beta)^{1/3}$ and $v = ((1-\alpha)(1-\beta))^{1/3}$, where β has degree x over α in signature 3. The following table lists the relation for the first few cases:

x	algebraic relations between u and v
2	$u + v - 1$
5	$(u + v - 1)^2 - 9uv$
8	$(u + v - 1)^4 + 9uv(4u + 4v + 5)(u + v - 1) - 162u^2v^2$
11	$u + v + 6\sqrt{uv} + 3\sqrt{3}\sqrt[4]{uv}(\sqrt{u} + \sqrt{v}) - 1$

Proof. First notice (26) simplifies to

$$xH\left(\frac{x}{3}\right) = -4\pi \int_0^1 \log \frac{1-(1-\beta)^{1/3}}{\beta^{1/3}} d \arctan \frac{1+2(1-\alpha)^{1/3}}{\sqrt{3}}.$$

If we set $x = 1$, then $\alpha = \beta$, and we obtain an integral for $H(1/3)$. Add the two formulas together, and notice that

$$\log \frac{(1-(1-\alpha)^{1/3})(1-(1-\beta)^{1/3})}{(\alpha\beta)^{1/3}} = \log\left(\frac{1-R+v}{u}\right),$$

where $u = (\alpha\beta)^{1/3}$, $v = ((1-\alpha)(1-\beta))^{1/3}$, and $R = (1-\alpha)^{1/3} + (1-\beta)^{1/3}$. Notice that $R^3 - 3vR - v^3 = 1 - u^3$. The identity becomes

$$xH\left(\frac{x}{3}\right) + H\left(\frac{1}{3}\right) = -4\pi \int_{\alpha \in [0,1]} \log\left(\frac{1-R+v}{u}\right) d \arctan \frac{1+2(1-\alpha)^{1/3}}{\sqrt{3}}.$$

The transformation $x \mapsto 1/x$ swaps α and β in the integral. The limits of integration are unchanged, because $\alpha = 0$ when $\beta = 0$, and $\alpha = 1$ when $\beta = 1$. The function inside the logarithm is unchanged, because it is symmetric in α and β . The integral becomes

$$\frac{1}{x}H\left(\frac{1}{3x}\right) + H\left(\frac{1}{3}\right) = -4\pi \int_{\alpha \in [0,1]} \log\left(\frac{1-R+v}{u}\right) d \arctan \frac{1+2(1-\beta)^{1/3}}{\sqrt{3}}.$$

Now add the formulas for $H(x/3)$ and $H(1/(3x))$, and use the addition formula for $\arctan z$ to complete the proof of (27). Notice that $v \in [1, 0]$ when $\alpha \in [0, 1]$, and v is monotone, therefore we can express the limits of integration in terms of v .

The specific algebraic relations between u and v are equivalent to modular equations in Ramanujan's theory of signature 3. The second degree modular equation [1, pg. 120, Theorem 7.1] shows that

$$(\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} = 1,$$

which is equivalent to $u + v - 1 = 0$. The fifth degree modular equation [1, pg. 124, Theorem 7.6] shows that

$$(\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} + 3(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 1;$$

this is equivalent to $(u + v - 1)^2 - 9uv = 0$. Finally, cases $x = 8$ and $x = 11$ follow from [1, pg. 132, Theorem 7.11] and [1, pg. 126, Theorem 7.8], respectively. \square

4. SIMPLIFICATION FOR $x = 2$

Before we attack the conductor 15 conjecture, we will briefly examine the much easier case when $x = 2$. Notice that we have already evaluated the left-hand side of (27) in formula (12). In fact, the following analysis will recover the correct identity. When $x = 2$ the relations between u , v , and R are given by

$$R^3 - 3vR - v^3 = 1 - u^3, \quad u + v = 1.$$

Therefore we have a genus 0 curve relating v and R :

$$R^3 - 3vR - v^3 = 1 - (1-v)^3.$$

Maple produces the following rational parameterizations:

$$R = \frac{(t-1)(2t^2-t+2)}{t^3+2}, \quad v = \frac{(t-1)^3}{t^3+2}.$$

If $v \in [0, 1]$, then $t \in [1, \infty)$. Therefore the integral becomes

$$\begin{aligned} 2H\left(\frac{2}{3}\right) + \frac{1}{2}H\left(\frac{1}{6}\right) + 2H\left(\frac{1}{3}\right) &= 4\pi \int_1^\infty \log \frac{1}{1-t+t^2} d \arctan \frac{\sqrt{3}t}{2-t} \\ &= -2\pi\sqrt{3} \int_1^\infty \frac{\log(1-t+t^2)}{1-t+t^2} dt \\ &= -\frac{4\pi^2}{3} \log 3 - 2\pi\sqrt{3}L(\chi_{-3}, 2). \end{aligned}$$

Mathematica evaluated the final integral after we made the substitution $t = (1 + \sqrt{3}\tan u)/2$. We can eliminate $H(1/3)$ by appealing to (14), and the identity finally reduces to (12).

5. SIMPLIFICATION FOR $x = 5$

Now we will find a formula for the conductor 15 elliptic curve. The ultimate goal of the following discussion is to obtain Proposition 3 and formula (32) below.

When $x = 5$ we can use (27), (14) and (16) to write

$$45L(E_{15}, 2) + 2\pi\sqrt{3}L(\chi_{-3}, 2) + \frac{4\pi^2}{3} \log 3 = -4\pi \int_{v \in [0,1]} \log x d \arctan(\sqrt{3}y),$$

where

$$x = \frac{1 - R + v}{u}, \quad y = \frac{1 + R}{1 - R - 2v}.$$

The algebraic relations between u , v , and R are given by

$$R^3 - 3vR - v^3 = 1 - u^3, \quad (u + v - 1)^2 - 9uv = 0.$$

Eliminating u , v , and R with successive resultants leads to a relation between x and y :

$$\begin{aligned} 0 &= (1 + x + x^2)(1 - 15x + 9x^2) + (4 + 20x - 12x^2)y \\ &\quad + (6 - 44x - 18x^2 - 36x^3 + 54x^4)y^2 + (4 + 60x - 36x^2)y^3 \\ &\quad + (1 - 9x + 9x^2)(1 + 3x + 9x^2)y^4. \end{aligned}$$

According to *Maple* this relation defines an elliptic curve. It is therefore possible to parameterize x and y by the Weierstrass coordinates of an elliptic curve. Assisted by *Maple's Weierstrassform* routine we discovered the following parametric formulas:

$$(28) \quad x = \frac{(1-t)^2(3t+t^2 - \sqrt{3}\sqrt{-3+t^2+2t^3})}{(3+t^2)^2},$$

$$(29) \quad y = \frac{(1+t)(3-6t-t^2 - 2\sqrt{3}\sqrt{-3+t^2+2t^3})}{(3-t)(3+t^2)}.$$

Notice that if $v \in [0, 1]$, then $x \in [1, 0]$ and $t \in (\infty, 1]$. We have the following proposition.

Proposition 3. *The following formula is true:*

$$(30) \quad \begin{aligned} & 45L(E_{15}, 2) + 2\pi\sqrt{3}L(\chi_{-3}, 2) + \frac{4\pi^2}{3} \log(3) \\ &= 4\pi \int_1^\infty \log \frac{(1-t)^2(3t+t^2 - \sqrt{3}\sqrt{-3+t^2+2t^3})}{(3+t^2)^2} \\ & \quad \times d \arctan \frac{\sqrt{3}(1+t)(3-6t-t^2 - 2\sqrt{3}\sqrt{-3+t^2+2t^3})}{(3-t)(3+t^2)}. \end{aligned}$$

Despite the fact that (30) is easy to compute numerically, it is still too complicated in its present form. The PSLQ algorithm was instrumental in discovering the following steps. First notice that the differential splits into two pieces. The following identity is trivial to verify with a computer:

$$d \arctan(\sqrt{3}y) = 2d \arctan \frac{t}{\sqrt{3}} + d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}}.$$

Furthermore, if we introduce the real Galois conjugate of x ,

$$(31) \quad \bar{x} = \frac{(1-t)^2(3t+t^2 + \sqrt{3}\sqrt{-3+t^2+2t^3})}{(3+t^2)^2},$$

then (30) can be broken into four integrals. We have

$$(32) \quad \begin{aligned} & 45L(E_{15}, 2) + 2\pi\sqrt{3}L(\chi_{-3}, 2) + \frac{4\pi^2}{3} \log(3) \\ &= 4\pi \int_1^\infty \log(x\bar{x}) d \arctan \frac{t}{\sqrt{3}} + 4\pi \int_1^\infty \log \frac{x}{\bar{x}} d \arctan \frac{t}{\sqrt{3}} \\ & \quad + 2\pi \int_1^\infty \log(x\bar{x}) d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}} \\ & \quad + 2\pi \int_1^\infty \log \frac{x}{\bar{x}} d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}}, \end{aligned}$$

where x and \bar{x} are defined in (28) and (31). It is unfortunate that the integrals in (32) are so complicated. We will simplify all four integrals in the following four lemmas. Two of them reduce to the desired quantities almost immediately.

Lemma 2. *The following evaluation holds:*

$$(33) \quad \int_1^\infty \log(x\bar{x}) d \arctan \frac{t}{\sqrt{3}} = -\sqrt{3}L(\chi_{-3}, 2) - \frac{2\pi}{3} \log 3.$$

Proof. First set $t = \sqrt{3} \tan \theta$, and notice

$$(34) \quad x\bar{x} = \frac{(1-t)^4}{(3+t^2)^2} = \frac{16}{9} \sin^4\left(\theta - \frac{\pi}{6}\right),$$

$$(35) \quad x + \bar{x} = \frac{2t(1-t)^2(3+t)}{(3+t^2)^2} = \frac{16}{3} \sin^2\left(\theta - \frac{\pi}{6}\right) \cos\left(\theta - \frac{\pi}{6}\right) \sin \theta.$$

It follows immediately that

$$\begin{aligned} \int_1^\infty \log(x\bar{x}) \, d \arctan \frac{t}{\sqrt{3}} &= \int_{\pi/6}^{\pi/2} \log\left(\frac{16}{9} \sin^4\left(\theta - \frac{\pi}{6}\right)\right) d\theta \\ &= 4 \int_0^{\pi/3} \log(2 \sin \theta) \, d\theta - \frac{2\pi}{3} \log 3 \\ &= -\sqrt{3}L(\chi_{-3}, 2) - \frac{2\pi}{3} \log 3, \end{aligned}$$

where the last step makes use of standard evaluations of the Clausen functions. \square

Lemma 3. *We have*

$$(36) \quad \int_1^\infty \log \frac{x}{\bar{x}} \, d \arctan \frac{t}{\sqrt{3}} = -2\pi \, \mathfrak{m}\left(1 + X + \frac{1}{X} + Y + \frac{1}{Y}\right).$$

Proof. Notice that $\bar{x}/x > 1$ and $0 < x/\bar{x} < 1$, whenever $t \in [1, \infty)$. Therefore, if $t = \sqrt{3} \tan \theta$ and $\theta \in [\pi/6, \pi/2]$, by Jensen's formula

$$\begin{aligned} \log \frac{x}{\bar{x}} &= -\mathfrak{m}\left(\left(Z - \frac{x}{\bar{x}}\right)\left(Z - \frac{\bar{x}}{x}\right)\right) \\ &= -\mathfrak{m}\left((Z+1)^2 - 16Z \cos^2\left(\theta - \frac{\pi}{6}\right) \sin^2 \theta\right), \end{aligned}$$

where we simplified the polynomial using (34) and (35). Also observe that if $\theta \in [0, \pi/6] \cup [\pi/2, \pi]$, then $|x/\bar{x}| = |\bar{x}/x| = 1$. In those cases the Mahler measure is identically zero. Therefore, we can write

$$\int_1^\infty \log \frac{x}{\bar{x}} \, d \arctan \frac{t}{\sqrt{3}} = - \int_0^\pi \mathfrak{m}\left((Z+1)^2 - 16Z \cos^2\left(\theta - \frac{\pi}{6}\right) \sin^2 \theta\right) d\theta$$

(introducing the notation $T = e^{i\theta}$ and $\zeta = e^{-\pi i/6}$)

$$\begin{aligned} &= - \int_0^\pi \mathfrak{m}\left((Z+1)^2 + Z(T^2\zeta - T^{-2}\zeta^{-1} + i)^2\right) d\theta \\ &= -\frac{1}{2} \int_0^{2\pi} \mathfrak{m}\left((Z+1)^2 + Z(T\zeta - T^{-1}\zeta^{-1} + i)^2\right) d\theta \\ &= -\pi \mathfrak{m}\left((Z+1)^2 + Z(T\zeta - T^{-1}\zeta^{-1} + i)^2\right) \end{aligned}$$

(using the substitution $(Z, T) \mapsto (X^2, i\zeta^{-1}Y)$ and the elementary properties of Mahler measures)

$$\begin{aligned} &= -\pi \mathfrak{m}\left((X^2+1)^2 - X^2(Y+Y^{-1}+1)^2\right) \\ &= -2\pi \mathfrak{m}\left(1 + X + X^{-1} + Y + Y^{-1}\right). \end{aligned}$$

\square

Lemma 4. *The following formula is true:*

$$(37) \quad \int_1^\infty \log \frac{x}{\bar{x}} d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}} \\ = 2\pi m(-Y^2 + X(1-Y - 2Y^2 - Y^3 + Y^4) - X^2Y^2).$$

Proof. The proof follows from parameterizing the integral differently. If we let

$$u := \sqrt{\frac{x}{\bar{x}}} = \sqrt{\frac{3t+t^2 - \sqrt{3(-3+t^2+2t^3)}}{3t+t^2 + \sqrt{3(-3+t^2+2t^3)}}}$$

and

$$v := \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}},$$

then

$$v = \pm \frac{1+u}{1-u} \sqrt{\frac{-1+3u-u^2}{1+u+u^2}};$$

the plus sign is chosen for $t \in [1, 3]$, and the minus sign is chosen for $t \in [3, \infty)$. Furthermore, when $t \in [1, 3]$ we have $u \in [1, (3 - \sqrt{5})/2]$, and when $t \in [3, \infty)$ we have $u \in [(3 - \sqrt{5})/2, 1]$. With a little work the integral becomes

$$\int_1^\infty \log \frac{x}{\bar{x}} d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}} \\ = 4 \int_1^{(3-\sqrt{5})/2} \log u d \arctan \left(\frac{1+u}{1-u} \sqrt{\frac{-1+3u-u^2}{1+u+u^2}} \right)$$

(taking r for $(-1+3u-u^2)/(1+u+u^2)$)

$$= 4 \int_{1/3}^0 \log \frac{3-r-\sqrt{5-14r-3r^2}}{2(1+r)} d \arctan \sqrt{\frac{r(5+r)}{1-3r}} \\ = 4 \int_0^{1/3} \log \frac{3-r+\sqrt{5-14r-3r^2}}{2(1+r)} d \arctan \sqrt{\frac{r(5+r)}{1-3r}}.$$

We can use Jensen's formula again, to substitute a one-variable Mahler measure for the logarithmic term:

$$= 4 \int_0^{1/3} m \left((1-Y)(1-Y^3) - \frac{4(1-3r)}{(1+r)^2} Y^2 \right) d \arctan \sqrt{\frac{r(5+r)}{1-3r}};$$

note that the polynomial

$$(1-Y)(1-Y^3) - \frac{4(1-3r)}{(1+r)^2} Y^2$$

has only one zero outside the unit circle for $r \in [0, 1/3]$. Finally, if $r(5+r)/(1-3r) = \tan^2 \theta$, then $(1-3r)/(1+r)^2 = \cos^2 \theta$ and the integral becomes

$$\begin{aligned} &= 4 \int_0^{\pi/2} \mathfrak{m}((1-Y)(1-Y^3) - 4Y^2 \cos^2 \theta) d\theta \\ &= \int_0^{2\pi} \mathfrak{m}((1-Y)(1-Y^3) - 4Y^2 \cos^2 \theta) d\theta \\ &= 2\pi \mathfrak{m}((1-Y)(1-Y^3)X - Y^2(X+1)^2), \end{aligned}$$

which expands into (37). □

Lemma 5. *The following formula is valid:*

$$(38) \quad \begin{aligned} &\int_1^\infty \log(x\bar{x}) d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}} \\ &= 2\pi \log 3 - 2 \int_0^1 \frac{(3+2u) \log u}{\sqrt{u(1-u)(3+u)(4+u)}} du. \end{aligned}$$

Proof. Let us begin by substituting (34) for $x\bar{x}$ and simplifying the differential. We have

$$\begin{aligned} &\int_1^\infty \log(x\bar{x}) d \arctan \frac{(3-t)(3+3t+2t^2)}{3(1+t)\sqrt{-3+t^2+2t^3}} \\ &= 3 \int_1^\infty \frac{1-4t-t^2}{(3+t^2)\sqrt{-3+t^2+2t^3}} \log \frac{(t-1)^2}{t^2+3} dt \end{aligned}$$

(after letting $t \mapsto (t+3)/(t-1)$)

$$= 3 \int_1^\infty \frac{1-4t-t^2}{(3+t^2)\sqrt{-3+t^2+2t^3}} \log \frac{4}{t^2+3} dt$$

(averaging the last two integrals)

$$= 3 \int_1^\infty \frac{1-4t-t^2}{(3+t^2)\sqrt{-3+t^2+2t^3}} \log \frac{2(t-1)}{t^2+3} dt.$$

If we let $u/3 = 2(t-1)/(t^2+3)$, then the integral splits into two parts for $t \in [1, 3]$ and $t \in [3, \infty)$. Some work reduces the entire expression to

$$\begin{aligned} &= -2 \int_0^1 \frac{(3+2u) \log(u/3)}{\sqrt{u(1-u)(3+u)(4+u)}} du \\ &= 2\pi \log 3 - 2 \int_0^1 \frac{(3+2u) \log u}{\sqrt{u(1-u)(3+u)(4+u)}} du, \end{aligned}$$

where on the final step we used the formula

$$\begin{aligned} \int_0^1 \frac{(3+2u) du}{\sqrt{u(1-u)(3+u)(4+u)}} &= \int_0^1 \frac{d(u(u+3))}{\sqrt{u(3+u)(4-u(3+u))}} \\ &= \int_0^4 \frac{dv}{\sqrt{v(4-v)}} = \pi. \end{aligned}$$

□

While formulas (33) and (36) have been reduced as far as possible, formulas (37) and (38) require slightly more attention.

The right-hand side of formula (37) is *extremely* surprising. The polynomial inside the Mahler measure is a knot invariant; namely,

$$A(X, Y) := -Y^2 + X(1 - Y - 2Y^2 - Y^3 + Y^4) - X^2Y^2$$

is the A-polynomial of the figure eight knot, denoted 4_1 by Rolfsen [19]. Boyd discussed this particular polynomial in great detail [6]. Its normalized Mahler measure, $\pi m(A)$, equals the volume of the hyperbolic manifold obtained from the complement of 4_1 in the 3-sphere. These volumes are well defined, and can be calculated in terms of values of the Bloch–Wigner dilogarithm [7]. The end result of that analysis is the following identity:

$$(39) \quad \pi m(-Y^2 + X(1 - Y - 2Y^2 - Y^3 + Y^4) - X^2Y^2) = \frac{3\sqrt{3}}{2} L(\chi_{-3}, 2).$$

Boyd has also informed us that Rodriguez-Villegas gave the first proof of this result. Although $A(X, Y) = 0$ defines a conductor 15 elliptic curve, we are at a loss to explain this surprising appearance of knot theory. We will speculate that it must be deeply connected to some type of underlying geometry associated with the signature 3 modular equations.

Formula (38) is an analogue of the integrals for the conductor 20 and 24 elliptic curves [21], and we will reduce it to a Mahler measure using a similar approach. For this, we introduce the integral

$$(40) \quad I(y) := -\frac{2}{\pi} \int_0^1 \frac{(y-1+2u) \log u}{\sqrt{u(1-u)(y-1+u)(y+u)}} du.$$

Proposition 4. *For $y \geq 1$, the following evaluation is valid:*

$$(41) \quad I(y) = 4 \log 2 - \frac{1}{8y^2} {}_4F_3\left(\frac{3}{2}, \frac{3}{2}, 1, 1 \mid \frac{1}{y^2}\right) - \frac{1}{y} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{1}{y^2}\right)$$

$$(42) \quad = m(4y) - m\left(\frac{4}{y}\right) - \log \frac{y}{4},$$

where $m(\alpha) = m(\alpha + X + X^{-1} + Y + Y^{-1})$.

Proof. First of all note that the integral $I(y)$ can be written as

$$I(y) = -\frac{2}{\pi} \int_0^1 \log u \cdot \frac{\partial w}{\partial u} du,$$

where

$$w(u) = w(u; y) := \arcsin \frac{2u(y-1+u) - y}{y}$$

and, for $y \geq 1$, the argument

$$\frac{2u(y-1+u) - y}{y}$$

monotonically changes from -1 to 1 when $u \in [0, 1]$. Since

$$\frac{\partial w}{\partial y} = \frac{1}{y} \frac{u(1-u)}{\sqrt{u(1-u)(y-1+u)(y+u)}},$$

the integration by parts for $y > 1$ results in

$$\begin{aligned} \frac{dI}{dy} &= -\frac{2}{\pi} \int_0^1 \log u \, d\left(\frac{\partial w}{\partial y}\right) \\ &= -\frac{2}{\pi} \log u \cdot \frac{\partial w}{\partial y} \Big|_{u=0}^{u=1} + \frac{2}{\pi} \int_0^1 \frac{\partial w}{\partial y} \frac{du}{u} \\ &= \frac{1}{\pi y} \int_0^1 \frac{2(1-u) \, du}{\sqrt{u(1-u)(y-1+u)(y+u)}}. \end{aligned}$$

Consider also the related integral

$$\frac{y+1}{\pi} \int_0^1 \frac{du}{\sqrt{u(1-u)(y-1+u)(y+u)}} = \frac{2}{\pi} K\left(\frac{2\sqrt{y}}{y+1}\right),$$

where

$$K(z) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \mid z^2\right), \quad |z| \leq 1,$$

is the complete elliptic integral of the first kind. On using the Gauss quadratic transformation

$$K(z) = \frac{1}{1+z} K\left(\frac{2\sqrt{z}}{1+z}\right)$$

with the choice $z = 1/y$ (hence $0 < z < 1$ for $y > 1$), we obtain

$$\begin{aligned} \frac{y+1}{\pi} \int_0^1 \frac{du}{\sqrt{u(1-u)(y-1+u)(y+u)}} &= \frac{2}{\pi} \cdot \frac{y+1}{y} K\left(\frac{1}{y}\right) \\ &= \frac{y+1}{y} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \mid \frac{1}{y^2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} (43) \quad \frac{y+1}{y} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \mid \frac{1}{y^2}\right) - y \frac{dI}{dy} &= \frac{1}{\pi} \int_0^1 \frac{(y-1+2u) \, du}{\sqrt{u(1-u)(y-1+u)(y+u)}} \\ &= \frac{1}{\pi} \int_0^1 \frac{\partial w}{\partial u} \, du = \frac{1}{\pi} (w(1; y) - w(0; y)) = 1, \end{aligned}$$

and so we have

$$\begin{aligned} y \frac{dI}{dy} &= -1 + \frac{y+1}{y} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| \frac{1}{y^2}\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} \frac{1}{y^{2n}} + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} \frac{1}{y^{2n+1}} \\ &= -y \frac{d}{dy} \left(\frac{1}{8y^2} {}_4F_3\left(\frac{3}{2}, \frac{3}{2}, 1, 1 \middle| \frac{1}{y^2}\right) + \frac{1}{y} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{y^2}\right) \right). \end{aligned}$$

The integration gives us

$$I(y) = C - \frac{1}{8y^2} {}_4F_3\left(\frac{3}{2}, \frac{3}{2}, 1, 1 \middle| \frac{1}{y^2}\right) - \frac{1}{y} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{y^2}\right).$$

To compute the constant of integration we use definition (40) of the integral $I(y)$:

$$C = \lim_{y \rightarrow +\infty} I(y) = -\frac{2}{\pi} \int_0^1 \frac{\log t}{\sqrt{t(1-t)}} dt = 4 \log 2.$$

Although we have done the computation for $y > 1$, the resulting formula (41) is valid for $y \geq 1$ because of continuity at $y = 1$. The two hypergeometric series can be further reduced to the Mahler measures by using the formulas [13], [18], [20]

$$m(\alpha) = \log \alpha - \frac{2}{\alpha^2} {}_4F_3\left(\frac{3}{2}, \frac{3}{2}, 1, 1 \middle| \frac{16}{\alpha^2}\right)$$

and

$$m(\alpha) = \frac{\alpha}{4} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{\alpha^2}{16}\right)$$

for $\alpha \geq 4$ and $0 < \alpha \leq 4$, respectively. This proves formula (42). \square

On invoking the computation in (43) we can also state formula (42) in the form

$$\frac{2}{\pi} \int_0^1 \frac{(y-1+2u) \log \frac{\sqrt{y}}{2u}}{\sqrt{u(1-u)(y-1+u)(y+u)}} du = m(4y) - m\left(\frac{4}{y}\right).$$

When $y = 4$ we obtain

$$\begin{aligned} (44) \quad -\frac{2}{\pi} \int_0^1 \frac{(3+2u) \log u}{\sqrt{u(1-u)(3+u)(4+u)}} du &= m(16) - m(1) = 10m(1) \\ &= 10m(1 + X + X^{-1} + Y + Y^{-1}), \end{aligned}$$

with the linear relation between $m(1)$ and $m(16)$ recently obtained by Lalín [14] (see also [10] for an elementary proof).

Combining (32), (33), (36), (37), (38), (39), and (44), we finally arrive at

Main theorem. *The following relation holds true:*

$$L(E_{15}, 2) = \frac{4\pi^2}{15} m(1 + X + X^{-1} + Y + Y^{-1}).$$

6. CONCLUSION

This work has raised a number of questions which are worth mentioning. The first is whether or not it is possible to say something about the L -functions of conductor 33 elliptic curves. Equations (22) and (27) can be used to produce a ‘coupled’ identity, relating $L(E_{11}, 2)$ and $L(E_{33}, 2)$ to an elementary integral. Unfortunately, the integral presents an enormous obstacle. The polynomial relating v and R (obtained from eliminating u in (27)) has genus 3. `Maple` failed to find parametric formulas, and our analysis stalled. As a final complication, Boyd’s paper does not mention any identities involving conductor 33 L -series [5]. It seems that an additional method of evaluating (27) is needed.

It is also worth understanding why our method produces so many ‘coupled’ identities. Perhaps one explanation for the conductor 11–33 pair, is that they both arise from integrating modular forms on the same congruence subgroup, $\Gamma_0(33)$. We have also produced a massively complicated formula for the conductor 24–48 pair, which may have a similar justification. The final puzzling aspect of this work is that our proof of the conductor 15 case required the non-trivial evaluation (39) of the Mahler measure of a knot polynomial. The fact that the conductor 15 L -series couples to a Dirichlet L -series, was probably fortunate for our calculations.

Acknowledgements. The authors would like to thank David Boyd and Nathan Dunfield for their valuable comments and details on identity (39). We kindly acknowledge Bruce Berndt, David Boyd, Christopher Deninger, and Michael Somos for their useful comments and encouragement.

REFERENCES

- [1] B. C. BERNDT, *Ramanujan’s Notebooks, Part V* (Springer-Verlag, New York, 1998).
- [2] M. J. BERTIN, Mesure de Mahler d’une famille de polynômes, *J. Reine Angew. Math.* **569** (2004), 175–188.
- [3] J. M. BORWEIN and P. B. BORWEIN, A cubic counterpart of Jacobi’s identity and the AGM, *Trans. Amer. Math. Soc.* **323** (1991), 691–701.
- [4] J. M. BORWEIN, P. B. BORWEIN, and F. GARVAN, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.* **343** (1994), 35–47.
- [5] D. W. BOYD, Mahler’s measure and special values of L -functions, *Experiment. Math.* **7** (1998), 37–82.
- [6] D. W. BOYD, Mahler’s measure and invariants of hyperbolic manifolds, in: *Number theory for the millennium*, M. A. Bennett et al., eds. (A K Peters, Boston, 2002), 127–143.
- [7] D. W. BOYD and F. RODRIGUEZ-VILLEGAS, Mahler’s measure and the dilogarithm (II), with an appendix by N. M. Dunfield, preprint [arXiv: math/0308041](https://arxiv.org/abs/math/0308041) [[math.NT](https://arxiv.org/abs/math.NT)] (2003).
- [8] F. BRUNAULT, Version explicite du théorème de Beilinson pour la courbe modulaire $X_1(N)$, *C. R. Math. Acad. Sci. Paris* **343** (2006), no. 8, 505–510.
- [9] C. DENINGER, Deligne periods of mixed motives, K -theory and the entropy of certain \mathbb{Z}^n -actions, *J. Amer. Math. Soc.* **10** (1997), no. 2, 259–281.
- [10] J. GUILLERA and M. ROGERS, Mahler measure and the WZ algorithm, preprint [arXiv: 1006.1654](https://arxiv.org/abs/1006.1654) [[math.NT](https://arxiv.org/abs/math.NT)] (2010).
- [11] G. KÖHLER, *Eta products and theta series identities* (Springer-Verlag, Heidelberg, 2011).
- [12] M. KONTSEVICH and D. ZAGIER, Periods, in: *Mathematics unlimited — 2001 and beyond* (Springer, Berlin, 2001), 771–808.

- [13] N. KUROKAWA and H. OCHIAI, Mahler measures via crystalization, *Comment. Math. Univ. St. Pauli* **54** (2005), 121–137.
- [14] M. N. LALÍN, On a conjecture by Boyd, *Int. J. Number Theory* **6** (2010), no. 3, 705–711.
- [15] Y. MARTIN and K. ONO, Eta-quotients and elliptic curves, *Proc. Amer. Math Soc.* **125** (1997), no. 11, 3169–3176.
- [16] A. MELLIT, Elliptic dilogarithms and parallel lines, preprint (2009).
- [17] D. PATHAKJEE, Z. ROSNBRICK, and E. YOONG, Elliptic curves, eta quotients and hypergeometric functions, preprint (2010).
- [18] F. RODRIGUEZ-VILLEGAS, Modular Mahler measures I, in: *Topics in number theory* (University Park, PA, 1997), Math. Appl. **467** (Kluwer Acad. Publ., Dordrecht, 1999), 17–48.
- [19] D. ROLFSEN, *Knots and links* (Publish or Perish, Berkeley, 1976).
- [20] M. ROGERS, Hypergeometric formulas for lattice sums and Mahler measures, *Intern. Math. Res. Not.* (to appear), preprint [arXiv: 0806.3590](https://arxiv.org/abs/0806.3590) [math.NT] (2008).
- [21] M. ROGERS and W. ZUDILIN, From L -series of elliptic curves to Mahler measures, preprint [arXiv: 1012.3036](https://arxiv.org/abs/1012.3036) [math.NT] (2010).
- [22] C. J. SMYTH, On measures of polynomials in several variables, *Bull. Austral. Math. Soc.* **23** (1981), no. 1, 49–63.
- [23] M. SOMOS, Dedekind eta function product identities, available at <http://eta.math.georgetown.edu/>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, USA

E-mail address: mathewrogers@gmail.com

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES, THE UNIVERSITY OF NEWCASTLE,
CALLAGHAN, NSW 2308, AUSTRALIA

E-mail address: wadim.zudilin@newcastle.edu.au