

IDENTITIES FOR THE RAMANUJAN ZETA FUNCTION

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ABSTRACT. We prove formulas for special values of the Ramanujan tau zeta function. Our formulas show that $L(\Delta, k)$ is a period in the sense of Kontsevich and Zagier when $k \geq 12$. As an illustration, we reduce $L(\Delta, k)$ to explicit integrals of hypergeometric and algebraic functions when $k \in \{12, 13, 14, 15\}$.

1. INTRODUCTION

1.1. **Background and previous results.** Ramanujan introduced his zeta function in 1916 [11]. Following Ramanujan, let

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $q = e^{2\pi iz}$. Ramanujan observed that $\tau(n)$ is multiplicative, and this led him to study the Dirichlet series:

$$L(\Delta, s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

Mordell subsequently proved that $L(\Delta, s)$ has an Euler product, and satisfies the functional equation $(2\pi)^{s-12} \Gamma(12-s) L(\Delta, 12-s) = (2\pi)^{-s} \Gamma(s) L(\Delta, s)$ [9], [20, pg. 242]. The main goal of this paper is to prove formulas for $L(\Delta, k)$ when k is a positive integer.

Kontsevich and Zagier defined a *period* to be a number which can be expressed as a multiple integral of algebraic functions, over a domain described by algebraic equations [8]. The ring of periods contains both the algebraic numbers, and certain transcendental numbers like π and $\log 2$. It follows from the work of Beilinson [2], and Deninger and Scholl [7], that special values of L -functions attached to modular forms are also periods. Paraphrasing [8, p. 24], their results follow from deep cohomological manipulations, and a careful study of values of regulators.

Following Deligne, we say that $L(\Delta, k)$ is a critical L -value if $1 \leq k \leq 11$. Kontsevich and Zagier summarized the properties of critical L -values in [8]. It is relatively easy to relate these values to integrals of algebraic functions. The standard Mellin

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transform gives

$$L(\Delta, k) = \frac{(2\pi)^k}{(k-1)!} \int_0^\infty u^{k-1} \Delta(iu) du \quad (1)$$

whenever $k \geq 1$. The usual method for obtaining an elementary integral, is to set

$$u = \frac{F(1-\alpha)}{2F(\alpha)}, \quad (2)$$

where $F(\alpha)$ is the classical hypergeometric series:

$$F(\alpha) := \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{\alpha}{16}\right)^n \quad (3)$$

$$= \frac{2}{\pi} \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\alpha u^2)}}. \quad (4)$$

Notice that $2iu$ is the period ratio of the elliptic curve $y^2 = (1-x^2)(1-\alpha x^2)$. Now appeal to the classical formulas [3, pg. 124], [3, pg. 120]:

$$\Delta(iu) = \frac{1}{16} \alpha (1-\alpha)^4 [F(\alpha)]^{12}, \quad (5)$$

$$\frac{du}{d\alpha} = \frac{-1}{2\pi\alpha(1-\alpha)[F(\alpha)]^2}, \quad (6)$$

and notice that $\alpha \in (1, 0)$ when $u \in (0, \infty)$. Equation (1) reduces to

$$L(\Delta, k) = \frac{\pi^{k-1}}{16(k-1)!} \int_0^1 (1-\alpha)^3 [F(\alpha)]^{11-k} [F(1-\alpha)]^{k-1} d\alpha. \quad (7)$$

Substituting (4) allows us to deduce that $\pi^{11-k} L(\Delta, k)$ is a period if $1 \leq k \leq 11$. Kontsevich and Zagier illustrated this point with an equivalent identity [8, pg. 24].

1.2. Main results. It is not at all obvious that $L(\Delta, k)$ is a period if $k \geq 12$. The method from the previous section fails, because the integrand in equation (7) becomes a ratio of algebraic and hypergeometric functions. It seems to be very difficult to pass from equation (7) to anything interesting. In this paper we will use ideas from the philosophy established jointly with Zudilin [16], [17], [22], to reduce these L -values to integrals of algebraic and hypergeometric functions. For example, we prove the following theorem:

Theorem 1. *The following identity is true:*

$$L(\Delta, 12) = -\frac{128\pi^{11}}{8241 \cdot 11!} \int_0^1 [F(\alpha)F(1-\alpha)]^5 \times \left(\frac{2 + 251\alpha + 876\alpha^2 + 251\alpha^3 + 2\alpha^4}{1-\alpha} \right) \log \alpha \, d\alpha, \quad (8)$$

where $F(\alpha)$ is defined in (3).

Formula (8) shows that $L(\Delta, 12)$ belongs to the ring of periods. Most of our identities involve hypergeometric functions, but these always reduce to elliptic integrals by (4). Equation (8) becomes a massive twelve dimensional integral:

$$L(\Delta, 12) = \frac{512\pi}{1284977925} \int_0^1 \left[\int_0^1 \int_0^1 \frac{du \, dz}{\sqrt{(1-u^2)(1-z^2)(1-\alpha u^2)(1-(1-\alpha)z^2)}} \right]^5 \\ \times \left(\frac{2 + 251\alpha + 876\alpha^2 + 251\alpha^3 + 2\alpha^4}{1-\alpha} \right) \int_\alpha^1 \frac{1}{t} dt \, d\alpha. \quad (9)$$

Theorem 2 provides the key formulas we need to express $L(\Delta, k)$ in terms of two dimensional integrals of algebraic and hypergeometric functions for $k \geq 12$. Corollary 1 highlights examples when $k \in \{13, 14, 15\}$. The $k = 12$ case is apparently the only instance where a reduction to a one-dimensional integral is possible, and we discuss this case separately in Section 2. Finally, we note that equation (8) is closely related to the *moments of elliptic integrals* studied in [1], [19], and [21].

2. A FORMULA FOR $L(\Delta, 12)$

The main goal of this section is to prove a formula for $L(\Delta, 12)$ using the method developed in [16] and [17]. In Section 3 we study $L(\Delta, k)$ for arbitrary $k \geq 12$.

The crucial first step is to decompose $\Delta(z)$ into a linear combination of products of two Eisenstein series. The usual Eisenstein series is defined by

$$E_k(z) := 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \frac{n^{k-1} e^{2\pi i n z}}{1 - e^{2\pi i n z}}.$$

The most famous decomposition of $\Delta(z)$ is due to Ramanujan:

$$1728\Delta(z) = E_4(z)^3 - E_6(z)^2, \quad (10)$$

but this formula involves $E_4(z)^3$, and the method from [16] and [17] only applies to modular forms which decompose into products of two Eisenstein series. We avoid this obstruction by considering a linear combination of Δ 's.

Lemma 1. *We have*

$$\Delta(z) + 24\Delta(2z) + 2^{11}\Delta(4z) = \frac{8}{504^2} [E_6(2z) - 64E_6(4z)] \\ \times [E_6(z) - 33E_6(2z) + 32E_6(4z)]. \quad (11)$$

Proof. It is easy to show that both sides of the equation are modular forms on $\Gamma_0(4)$. By the standard valence formula for congruence subgroups [18], the two sides are equal if their Fourier series expansions agree to more than $[\Gamma(1) : \Gamma_0(4)] = 6$ terms. We used a computer to check that the first 1000 Fourier coefficients agree, and thus we conclude that the identity is true. It is possible to construct an alternative elementary proof by applying formulas from Ramanujan's notebooks (use [3, pg. 124, Entry 12] and [3, pg. 126, Entry 13]). \square

Now apply an involution to the first term on the right-hand side of (11). The equation becomes

$$\Delta(z) + 24\Delta(2z) + 2^{11}\Delta(4z) = \frac{8}{504^2(2z)^6} \left[E_6\left(\frac{-1}{2z}\right) - E_6\left(\frac{-1}{4z}\right) \right] \times [E_6(z) - 33E_6(2z) + 32E_6(4z)].$$

Suppose that $z = iu$ with $u \geq 0$. The right-hand side reduces to a four-dimensional infinite series. We have

$$\Delta(iu) + 24\Delta(2iu) + 2^{11}\Delta(4iu) = \frac{1}{8u^6} \sum_{\substack{n,m,r,s \geq 1 \\ m,r,s \text{ odd}}} (nr)^5 e^{-2\pi\left(\frac{nm}{4u} + rsu\right)}.$$

Multiply both sides by u^{k-1} and integrate for $u \in (0, \infty)$. By uniform convergence:

$$8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) = \frac{(2\pi)^k}{(k-1)!} \sum_{\substack{n,m,r,s \geq 1 \\ m,r,s \text{ odd}}} (nr)^5 \int_0^\infty u^{k-7} e^{-2\pi\left(\frac{nm}{4u} + rsu\right)} du. \quad (12)$$

The rational term on the left cancels the Euler factor of $L(\Delta, s)$ at the prime $p = 2$. Notice that

$$(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) = \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \frac{\tau(n)}{n^s}.$$

Finally apply the key trick: *Use a change of variables to swap the indices of summation inside the integral in (12). If the trick is properly executed, then the right-hand side reduces to an integral involving modular functions.*

Proof of Theorem 1. Set $k = 12$ and let $u \mapsto nu/r$ in (12). The formula becomes

$$\begin{aligned} \frac{8241}{2^{22}\pi^{12}} L(\Delta, 12) &= \frac{1}{11!} \int_0^\infty u^5 \sum_{\substack{r,m \geq 1 \\ r,m \text{ odd}}} \frac{1}{r} e^{-\frac{2\pi rm}{4u}} \sum_{\substack{n,s \geq 1 \\ s \text{ odd}}} n^{11} e^{-2\pi nsu} du \\ &= \frac{1}{11!} \int_0^\infty u^5 \log \left(\prod_{\substack{m=1 \\ m \text{ odd}}}^\infty \frac{1 + e^{-\frac{2\pi m}{4u}}}{1 - e^{-\frac{2\pi m}{4u}}} \right) \sum_{n=1}^\infty \frac{n^{11} e^{-2\pi nu}}{1 - e^{-4\pi nu}} du. \end{aligned}$$

Now suppose that u and α are related by (2). In particular:

$$u = \frac{F(1-\alpha)}{2F(\alpha)}.$$

Then $\alpha \in (1, 0)$ when $u \in (0, \infty)$. We compute $du/d\alpha$ using (6), and various identities from Ramanujan's notebooks imply

$$\prod_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{1 + e^{-\frac{2\pi m}{4u}}}{1 - e^{-\frac{2\pi m}{4u}}} = \alpha^{-1/8}, \quad (13)$$

$$\sum_{n=1}^{\infty} \frac{n^{11} e^{-2\pi n u}}{1 - e^{-4\pi n u}} = \frac{1}{32} \alpha (2 + 251\alpha + 876\alpha^2 + 251\alpha^3 + 2\alpha^4) [F(\alpha)]^{12}. \quad (14)$$

We can prove (13) using [3, pg. 124, Entry 12], and the proof of (14) follows from the Eisenstein series relation $E_{12}(z) = E_6(z)^2$ and [3, pg. 126, Entry 13]. Combining the various formulas completes the proof. \square

3. FORMULAS FOR $L(\Delta, k)$ WHEN $k \geq 12$

In this section we obtain double integrals for $L(\Delta, k)$ when $k \geq 12$. In general, it seems to be difficult to simplify formulas for $L(f, k)$, when $k > \text{weight}(f)$. So far there is only one instance where such an L -value has been reduced to recognizable special functions. Zudilin proved that

$$\begin{aligned} \frac{768\sqrt{2}}{\pi^{3/2}} L(g, 3) = & \Gamma^2(1/4) {}_4F_3 \left(\begin{matrix} 1, 1, 1, \frac{1}{2} \\ \frac{7}{4}, \frac{3}{2}, \frac{3}{2} \end{matrix}; 1 \right) + 24\Gamma^2(3/4) {}_4F_3 \left(\begin{matrix} 1, 1, 1, \frac{1}{2} \\ \frac{5}{4}, \frac{3}{2}, \frac{3}{2} \end{matrix}; 1 \right) \\ & + 3\Gamma^2(1/4) {}_4F_3 \left(\begin{matrix} 1, 1, 1, \frac{1}{2} \\ \frac{3}{4}, \frac{3}{2}, \frac{3}{2} \end{matrix}; 1 \right). \end{aligned} \quad (15)$$

where $g(z) = \eta^2(4z)\eta^2(8z)$ is the weight 2 CM newform attached to conductor 32 elliptic curves [22], [23]. Rodriguez-Villegas and Boyd used numerical experiments to find many relations between Mahler measures and values of $L(f, k)$, but all of their conjectures are still open [5], [14].

Theorem 2. *Assume that $k \geq 12$. If k is odd:*

$$\begin{aligned} & 8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\ & = \frac{2^{k-7} i^{k-1} \pi^{2k-11} [\zeta(6-k)]^2}{(k-1)!(k-12)!} \int_0^\infty \int_u^\infty (z-u)^{k-12} P_{k-5}(u) Q_{k-5}(z) dz du, \end{aligned} \quad (16)$$

where

$$P_k(u) := E_k(iu) - (2 + 2^k)E_k(2iu) + 2^{k+1}E_k(4iu), \quad (17)$$

$$Q_k(z) := E_k(iz) - E_k(2iz). \quad (18)$$

If k is even:

$$\begin{aligned} & 8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\ & = -\frac{2^{k-1} i^k \pi^{2k-11} \zeta(1-k) \zeta(11-k)}{(k-1)!(k-12)!} \int_0^\infty \int_u^\infty (z-u)^{k-12} u^5 R_k(u) S_{k-10}(z) dz du, \end{aligned} \quad (19)$$

where

$$R_k(u) := E_k(2iu) - 2^k E_k(4iu), \quad (20)$$

$$S_k(z) := E_k(iz) - (1 + 2^{k-1})E_k(2iz) + 2^{k-1}E_k(4iz). \quad (21)$$

Proof. We prove (16) first. Assume that k is odd in (12), and let $u \mapsto mu/r$. The integral becomes

$$\begin{aligned} & 8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\ &= \frac{(2\pi)^k}{(k-1)!} \int_0^\infty u^{k-7} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{m^{k-6} e^{-2\pi mu}}{1 - e^{-4\pi mu}} \sum_{\substack{n, r \geq 1 \\ r \text{ odd}}} \frac{n^5}{r^{k-11}} e^{-\frac{2\pi nr}{4u}} du. \end{aligned}$$

Let $u \mapsto \frac{1}{4u}$, then

$$\begin{aligned} & 8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\ &= \frac{(2\pi)^k}{(k-1)!} \int_0^\infty 4(4u)^{5-k} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{m^{k-6} e^{-\frac{2\pi m}{4u}}}{1 - e^{-\frac{4\pi m}{4u}}} \sum_{\substack{n, r \geq 1 \\ r \text{ odd}}} \frac{n^5}{r^{k-11}} e^{-2\pi nru} du. \end{aligned} \quad (22)$$

Using the involution for $E_k(u)$, we easily find that

$$4(4u)^{5-k} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \frac{m^{k-6} e^{-\frac{2\pi m}{4u}}}{1 - e^{-\frac{4\pi m}{4u}}} = i^{k-1} 2^{5-k} \zeta(6-k) P_{k-5}(u), \quad (23)$$

where $P_k(u)$ is defined in (17). To simplify the second sum in (22), we require the integral:

$$\frac{e^{-2\pi au}}{a^s} = \frac{(2\pi)^s}{\Gamma(s)} \int_u^\infty (z-u)^{s-1} e^{-2\pi az} dz. \quad (24)$$

If $s \mapsto k-11$ and $a \mapsto nr$, then

$$\begin{aligned} \sum_{\substack{n, r \geq 1 \\ r \text{ odd}}} \frac{n^5}{r^{k-11}} e^{-2\pi nru} &= \sum_{\substack{n, r \geq 1 \\ r \text{ odd}}} \frac{n^{k-6}}{(nr)^{k-11}} e^{-2\pi nru} \\ &= \frac{(2\pi)^{k-11}}{(k-12)!} \int_u^\infty (z-u)^{k-12} \sum_{\substack{n, r \geq 1 \\ r \text{ odd}}} n^{k-6} e^{-2\pi nrz} dz \\ &= \frac{(2\pi)^{k-11} \zeta(6-k)}{2(k-12)!} \int_u^\infty (z-u)^{k-12} Q_{k-5}(z) dz, \end{aligned} \quad (25)$$

where $Q_k(z)$ is defined in (18). Finally combine (25), (23), and (22) to complete the proof of (16).

Next we prove (19). The steps are similar to the proof of (16), so we will be brief. Assume that k is even in equation (12), and let $u \mapsto nu/r$. Then we find

$$\begin{aligned}
& 8(1 + 24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\
&= \frac{(2\pi)^k}{(k-1)!} \int_0^\infty u^{k-7} \sum_{\substack{n,s \geq 1 \\ s \text{ odd}}} n^{k-1} e^{-2\pi n s u} \sum_{\substack{m,r \geq 1 \\ m,r \text{ odd}}} \frac{1}{r^{k-11}} e^{-\frac{2\pi m r}{4u}} du \\
&= \frac{(2\pi)^k}{(k-1)!} \int_0^\infty 4(4u)^{5-k} \sum_{\substack{n,s \geq 1 \\ s \text{ odd}}} n^{k-1} e^{-\frac{2\pi n s}{4u}} \sum_{\substack{m,r \geq 1 \\ m,r \text{ odd}}} \frac{1}{r^{k-11}} e^{-2\pi m r u} du, \\
&= \frac{(2\pi)^{2k-11}}{(k-1)!(k-12)!} \int_0^\infty \int_u^\infty (z-u)^{k-12} \left(4(4u)^{5-k} \sum_{\substack{n,s \geq 1 \\ s \text{ odd}}} n^{k-1} e^{-\frac{2\pi n s}{4u}} \right) \\
&\quad \times \left(\sum_{\substack{m,r \geq 1 \\ m,r \text{ odd}}} m^{k-11} e^{-2\pi m r z} \right) dz du.
\end{aligned}$$

The second equality follows from mapping $u \mapsto \frac{1}{4u}$, and the third equality follows from (24). Finally, it is easy to show that

$$\begin{aligned}
4(4u)^{5-k} \sum_{\substack{n,s \geq 1 \\ s \text{ odd}}} n^{k-1} e^{-\frac{2\pi n s}{4u}} &= -i^k 2^{11-k} \zeta(1-k) u^5 R_k(u), \\
\sum_{\substack{m,r \geq 1 \\ m,r \text{ odd}}} m^{k-11} e^{-2\pi m r z} &= \frac{1}{2} \zeta(11-k) S_{k-10}(z),
\end{aligned}$$

where $R_k(u)$ and $S_k(z)$ are defined in (20) and (21). □

Corollary 1. *Let $F(\alpha)$ denote the usual hypergeometric function, defined in (3). The following identities are true:*

$$L(\Delta, 13) = \frac{\pi^{13}}{q_{13}} \int_0^1 \int_0^\alpha [F(\alpha)F(1-\beta) - F(\beta)F(1-\alpha)] [F(\alpha)F(\beta)]^5 \\ \times \frac{(1+\alpha)(17-32\alpha+17\alpha^2)(2+13\beta+2\beta^2)}{\alpha(1-\beta)} d\beta d\alpha, \quad (26)$$

$$L(\Delta, 14) = \frac{\pi^{15}}{q_{14}} \int_0^1 \int_0^\alpha [F(\alpha)F(1-\beta) - F(\beta)F(1-\alpha)]^2 [F(\alpha)F(1-\alpha)]^5 \\ \times \frac{(2-\alpha)(5461-10922\alpha+5973\alpha^2-512\alpha^3+\alpha^4)(2-\beta)}{\alpha(1-\beta)} d\beta d\alpha, \quad (27)$$

$$L(\Delta, 15) = \frac{\pi^{17}}{q_{15}} \int_0^1 \int_0^\alpha [F(\alpha)F(1-\beta) - F(\beta)F(1-\alpha)]^3 [F(\alpha)F(\beta)]^5 \\ \times \frac{(31-47\alpha+33\alpha^2-47\alpha^3+31\alpha^4)(1+\beta)(1+29\beta+\beta^2)}{\alpha(1-\beta)} d\beta d\alpha, \quad (28)$$

where $q_{13} = 122987403000$, $q_{14} = 798232309875$, and $q_{15} = 67002093132975/4$.

Proof. We sketch the proof of equations (26) and (28) below. Set

$$u = \frac{F(1-\alpha)}{2F(\alpha)}, \quad z = \frac{F(1-\beta)}{2F(\beta)},$$

and then apply (6). Equation (16) becomes

$$8(1+24 \cdot 2^{-k} + 2^{11-2k}) L(\Delta, k) \\ = \frac{2^{k-7} i^{k-1} \pi^{2k-11} [\zeta(6-k)]^2}{(k-1)!(k-12)!} \\ \times \int_0^1 \int_0^\alpha \left(\frac{F(1-\beta)}{2F(\beta)} - \frac{F(1-\alpha)}{2F(\alpha)} \right)^{k-12} \frac{P_{k-5}(u)Q_{k-5}(z) d\beta d\alpha}{4\pi^2 \alpha(1-\alpha)\beta(1-\beta)F(\alpha)^2 F(\beta)^2}.$$

Finally we use the formulas

$$P_8(u) = 15(1-\alpha^2)(17-32\alpha+17\alpha^2)[F(\alpha)]^8, \\ Q_8(z) = 15\beta(2+13\beta+2\beta^2)[F(\beta)]^8, \\ P_{10}(u) = 33(1-\alpha)(31-47\alpha+33\alpha^2-47\alpha^3+31\alpha^4)[F(\alpha)]^{10}, \\ Q_{10}(z) = -\frac{33}{2}\beta(1+\beta)(1+29\beta+\beta^2)[F(\beta)]^{10},$$

when $k = 13$ and $k = 15$. These formulas are easy to prove by expressing $E_k(iu)$ in terms of polynomials in $E_4(iu)$ and $E_6(iu)$, and then appealing to [3, pg. 126, Entry 13]. In practice, we used numerical searches to find linear dependencies between

$$P_k(u), Q_k(u), \text{ and } \left\{ \alpha^j [F(\alpha)]^k \right\}_{j=0}^{j=k}. \quad \square$$

If u and α are related by (2), then it is a classical fact that $E_k(2^j iu) = (\text{polynomial in } \alpha) \times [F(\alpha)]^k$ for $j \in \{0, 1, 2\}$ and k even. This makes it possible to see that the pattern of Corollary 1 continues. The identities for $L(\Delta, k)$ are always two dimensional integrals containing

$$[F(\alpha)F(1 - \beta) - F(\beta)F(1 - \alpha)]^{k-12}.$$

Furthermore, if k is even the integral contains $[F(\alpha)F(1 - \alpha)]^5$, and if k is odd the integral contains $[F(\alpha)F(\beta)]^5$.

4. SPECULATION AND CONCLUSION

We have proved that $L(\Delta, k)$ is a period for $k \geq 12$. It is interesting to speculate on what simplifications might be possible for the integrals in Theorem 1 and Corollary 1. We can draw an analogy with the case of weight two modular forms. If we select a specific modular form such as $f(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$, then it is possible to prove results like

$$\frac{15}{4\pi^2} L(f, 2) = \int_0^1 \int_0^1 \log |1 + X + X^{-1} + Y + Y^{-1}| dt ds, \quad (29)$$

where $X = e^{2\pi it}$ and $Y = e^{2\pi is}$. The integral on the right is a Mahler measure, and the surface obtained from setting $1 + X + X^{-1} + Y + Y^{-1} = 0$, is precisely the elliptic curve attached to $f(z)$ by the modularity theorem. Rodriguez-Villegas gave a very nice explanation of *why* results like (29) exist [13], [6], [4], [17]. The key point for us, is that the proof of (29) *does not require any prior knowledge of the elliptic curve attached to $f(z)$* . Thus it seems plausible that there might be an analogous relation between $L(\Delta, 12)/\pi^{12}$ and the Mahler measure of a polynomial in 12 variables. If such an identity exists, then it should be possible to derive it from equation (8) with only calculus. Furthermore, the 12 variable polynomial should give an affine model of a hypersurface attached to $\Delta(z)$. Deligne found such a hypersurface in his proof of the Ramanujan-Petersson conjectures, but his hypersurface is typically described using étale cohomology.

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