Modular equations and lattice sums

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Abstract We highlight modular equations due to Ramanujan and Somos, and use them to prove new relations between lattice sums and hypergeometric functions. We also discuss progress towards solving Boyd's Mahler measure conjectures. Finally, we conjecture a new formula for $L(E,2)$ of conductor 17 elliptic curves.

1 Introduction

Modular equations appear in a variety of number-theoretic contexts. Their connection to formulas for $1/\pi$ [15], Ramanujan constants such as $e^{\pi\sqrt{163}}$ [22], and elliptic curve cryptography is well established. In the classical theory of modular forms, an nth degree modular equation is an algebraic relation between $j(\tau)$ and $j(n\tau)$, where $j(\tau)$ is the j-invariant. For our purposes a modular equation is simply a non-trivial algebraic relation between theta or eta functions. In this paper we use modular equations to study four-dimensional lattice sums. The lattice sums are interesting because they arise in the study of Mahler measures of elliptic curves.

There are many hypothetical relations between special values of L-series of elliptic curves, and Mahler measures of two-variable polynomials. The Mahler measures $m(\alpha)$, $n(\alpha)$, and $g(\alpha)$ are defined by

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$$m(\alpha) := \int_0^1 \int_0^1 \log\left|y+y^{-1}+z+z^{-1}+\alpha\right| \mathrm{d}\theta_1 \mathrm{d}\theta_2, \tag{1}$$

$$n(\alpha) := \int_0^1 \int_0^1 \log\left|y^3+z^3+1-\alpha yz\right| \mathrm{d}\theta_1 \mathrm{d}\theta_2, \tag{2}$$

$$g(\alpha) := \int_0^1 \int_0^1 \log\left|(y+1)(z+1)(y+z)-\alpha yz\right| \mathrm{d}\theta_1 \mathrm{d}\theta_2, \tag{3}$$

where $y = e^{2\pi i\theta_1}$, and $z = e^{2\pi i\theta_2}$. Boyd conjectured that for all integral values of $k \neq 4$ [6]:

$$m(k) \stackrel{?}{=} \frac{q}{\pi^2} L(E,2),$$

where E is an elliptic curve, q is rational, and both E and q depend on k. He also discovered many formulas involving $g(\alpha)$ and $n(\alpha)$. In cases where E has a small conductor, it is frequently possible to express $L(E,2)$ in terms of four-dimensional lattice sums. Thus many of Boyd's identities can be regarded as series acceleration formulas. The main goal of this paper is to prove new formulas for the lattice sum $F(b,c)$, defined in (13). There are at least 18 instances where $F(b,c)$ is known (or conjectured) to reduce to integrals of elementary functions. The modular equations of Ramanujan and Somos are the main tools in our analysis.

2 Eta function product identities

Somos discovered thousands of new modular equations by searching for linear relations between products of Dedekind eta functions. Somos refers to these formulas as *eta function product identities*. The existence of eta function product identities follows from the fact that $j(\tau)$ equals a rational expression involving eta functions. One can transform classical modular equations into eta function product identities by simply clearing denominators. Somos's experimental approach turned up many surprisingly simple identities. In order to give an example, first consider the eta function with respect to q:

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24},$$

and adopt the shorthand notation

$$e_j = \eta(q^j).$$

Formula (4) is the smallest eta function product identity in Somos's list [20]:

$$e_2 e_6 e_{10} e_{30} = e_1 e_{12} e_{15} e_{20} + e_3 e_4 e_5 e_{60}. \tag{4}$$

Notice that all three monomials are products of four eta functions, and are essentially weight-two modular forms. No identities are known if the eta products have weight less than two, and (4) appears to be the only three-term linear relation between products of four eta functions. Many additional identities are known if the number of terms is allowed to increase, or if eta products of higher weight are considered. For additional examples see formulas (16), (24), (25), (28), and (31) below.

Identities such as (4) can be proved almost effortlessly with the theory of modular forms. A typical proof involves checking that the first few Fourier coefficients of an identity vanish. Sturm's Theorem furnishes an upper bound on the number of coefficients that need to be examined [14]. We note that it is often possible, but usually more difficult, to prove such identities via q-series methods. Ramanujan derived hundreds of modular equations with q-series techniques. We conclude this section by proving (4). This proof fills in a gap in the literature of Ramanujan-Berndt-style proofs.

Theorem 1. *The identity* (4) *is true.*

Proof. Let us denote the usual theta functions by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{5}$$

Furthermore define u_j and z_j by

$$u_j := 1 - \frac{\varphi^4(-q^j)}{\varphi^4(q^j)}, \qquad z_j := \varphi^2(q^j).$$

Ramanujan uses a slightly different notation [2]. He typically sets $\alpha = u_1$, and says that "β has degree j over α" when $\beta = u_j$. Certain values of the eta function can be expressed in terms of u_1 and z_1 [2, p. 124]. We have

$$\eta(q) = 2^{-1/6} u_1^{1/24} (1-u_1)^{1/6} \sqrt{z_1}, \tag{6}$$

$$\eta(q^2) = 2^{-1/3} \{u_1(1-u_1)\}^{1/12} \sqrt{z_1}, \tag{7}$$

$$\eta(q^4) = 2^{-2/3} u_1^{1/6} (1-u_1)^{1/24} \sqrt{z_1}. \tag{8}$$

Now we prove (4). By (7) the left-hand side of the identity becomes

$$e_2 e_6 e_{10} e_{30} = 2^{-4/3} \{u_1 u_3 u_5 u_{15} (1-u_1)(1-u_3)(1-u_5)(1-u_{15})\}^{1/12} \sqrt{z_1 z_3 z_5 z_{15}}.$$

By (6) and (8), the right-hand side of the identity becomes

$$\begin{aligned} e_1 e_{12} e_{15} e_{20} + e_3 e_4 e_5 e_{60} \\ = 2^{-5/3} \Big(&\{u_3 u_5 (1-u_1)(1-u_{15})\}^{1/6} \{u_1 u_{15} (1-u_3)(1-u_5)\}^{1/24} \\ &+ \{u_1 u_{15} (1-u_3)(1-u_5)\}^{1/6} \{u_3 u_5 (1-u_1)(1-u_{15})\}^{1/24} \Big) \sqrt{z_1 z_3 z_5 z_{15}}. \end{aligned}$$

Combining the last two formulas shows that (4) is equivalent to

$$\begin{aligned} &2^{1/3}\left\{u_1u_3u_5u_{15}(1-u_1)(1-u_3)(1-u_5)(1-u_{15})\right\}^{1/24} \\ &\qquad = \left\{u_3u_5(1-u_1)(1-u_{15})\right\}^{1/8} + \left\{u_1u_{15}(1-u_3)(1-u_5)\right\}^{1/8}. \end{aligned} \tag{9}$$

It is sufficient to show that (9) can be deduced from Ramanujan's modular equations.

The first modular equation we require can be recovered by multiplying entries 11.1 and 11.2 in [2, p. 383]:

$$\left((u_1u_{15})^{1/8} + \left\{(1-u_1)(1-u_{15})\right\}^{1/8}\right)\left((u_3u_5)^{1/8} + \left\{(1-u_3)(1-u_5)\right\}^{1/8}\right) = 1.$$

Rearranging yields an identity for the right-hand side of (9):

$$\begin{aligned} &\left\{u_3u_5(1-u_1)(1-u_{15})\right\}^{1/8} + \left\{u_1u_{15}(1-u_3)(1-u_5)\right\}^{1/8} \\ &\qquad = 1 - \left\{u_1u_3u_5u_{15}\right\}^{1/8} - \left\{(1-u_1)(1-u_3)(1-u_5)(1-u_{15})\right\}^{1/8}. \end{aligned} \tag{10}$$

By [2, p. 385, Entry 11.14], it is clear that

$$\begin{aligned} &1 - \left\{u_1u_3u_5u_{15}\right\}^{1/8} - \left\{(1-u_1)(1-u_3)(1-u_5)(1-u_{15})\right\}^{1/8} \\ &\qquad = 2^{1/3}\left\{u_1u_3u_5u_{15}(1-u_1)(1-u_3)(1-u_5)(1-u_{15})\right\}^{1/24}. \end{aligned} \tag{11}$$

The theorem follows from combining (10) and (11) to recover (9).

We find it slightly surprising that Ramanujan overlooked (4). He possessed a tremendous ability to derive modular equations, and he discovered all of the necessary intermediate results. Perhaps it is simply not obvious *why* identities such as (4) exist. We are unable to offer much insight, beyond pointing out that there are many additional formulas in Somos's tables. A conceptual proof of (4) might lead to a systematic method for generating more identities. In the next section, we demonstrate that this is an important topic in the study of lattice sums.

3 Lattice Sums

In this section we investigate four-dimensional lattice sums. Many of these sums are related to L-functions of elliptic curves. Let us define

$$\begin{aligned} F(a,b,c,d) := &(a+b+c+d)^2 \\ &\times \sum_{n_i=-\infty}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left(a(6n_1+1)^2+b(6n_2+1)^2+c(6n_3+1)^2+d(6n_4+1)^2\right)^2}. \end{aligned}$$

The four-dimensional series is not absolutely convergent, so it is necessary to employ summation by cubes [5]. Notice that Euler's pentagonal number theorem can be used to represent $F(a,b,c,d)$ as an integral

$$F(a,b,c,d) = -\frac{(a+b+c+d)^2}{24^2}\int_0^1 \eta(q^a)\eta(q^b)\eta(q^c)\eta(q^d)\log q\frac{\mathrm{d}q}{q}. \tag{12}$$

We also use the shorthand notation

$$F(b,c) := F(1,b,c,bc), \tag{13}$$

since we are primarily interested in cases where $a = 1$, $d = bc$, and b and c are rational.

The interplay between values of $F(b,c)$, Boyd's Mahler measure conjectures, and the Beilinson conjectures is outlined in [17]. If $(b,c) \in \mathbb{N}^2$ and $(1+b)(1+c)$ divides 24, then $F(b,c) = L(E,2)$ for an elliptic curve E. Formulas are now proved relating each of those eight L-values to Mahler measures [23]. Mahler measures often reduce to generalized hypergeometric functions, so many of Boyd's identities can be regarded as series transformations [16], [13]. It is known that

$$\begin{aligned}
m(\alpha) =& \mathrm{Re}\left[\log(\alpha) - \frac{2}{\alpha^2}{}_4F_3\left({\tfrac{3}{2},\tfrac{3}{2},1,1 \atop 2,2,2};\frac{16}{\alpha^2}\right)\right], \text{if } \alpha \neq 0,\\
n(\alpha) =& \mathrm{Re}\left[\log(\alpha) - \frac{2}{\alpha^3}{}_4F_3\left({\tfrac{4}{3},\tfrac{5}{3},1,1 \atop 2,2,2};\frac{27}{\alpha^3}\right)\right], \text{ if } |\alpha| \text{ is sufficiently large,}\\
3g(\alpha) =& n\left(\frac{\alpha+4}{\alpha^{2/3}}\right) + 4n\left(\frac{\alpha-2}{\alpha^{1/3}}\right), \text{ if } |\alpha| \text{ is sufficiently large.}
\end{aligned}$$

The function $m(\alpha)$ also reduces to a ${}_3F_2$ function if $\alpha \in \mathbb{R}$ [12], [17]. Rogers and Zudilin [18] recently proved that

$$F(3,5) = \frac{4\pi^2}{15}m(1) = \frac{\pi^2}{15}{}_3F_2\left({\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2} \atop 1,\tfrac{3}{2}}\middle|\frac{1}{16}\right). \tag{14}$$

Equation (14) is equivalent to a formula that Deninger conjectured [9]. The same formula helped motivate Boyd's seminal paper [6]. It is also possible to prove formulas for values such as $F(1,4)$ and $F(2,2)$ [17]. These lattice sums are not related to elliptic curve L-values in an obvious way, so it was conjectured that it should be possible to "sum up" $F(b,c)$ for arbitrary values of b and c.

3.1 Lacunary cases

In general, the difficulty of dealing with a lattice sum depends on whether it is *lacunary* or *non-lacunary*. Lacunary examples are usually much easier to work with. We say that a lattice sum is lacunary if it equals the Mellin transform of a lacunary modular form. Modular forms are called lacunary whenever their Fourier series coefficients have zero arithmetic density. To detect lacunary eta products, first expand the eta product in a series

$$\eta(q^a)\eta(q^b)\eta(q^c)\eta(q^d) = q^{(a+b+c+d)/24}\left(a_0 + a_1 q + a_2 q^2 + \dots\right), \qquad (15)$$

and then check that $a_n = 0$ for almost all n. It seems to be an open problem to classify quadruples (a,b,c,d) which make (15) lacunary. While cusp forms associated with CM elliptic curves are always lacunary, only three of those cusp forms actually equal products of four eta functions [14]. Less is known if an eta product is not obviously related to an elliptic curve. Empirically, it appears that many values of $e_a e_b e_c e_d$, can be expressed as linear combinations of two-dimensional theta functions. Expansions such as

$$\eta^2(q)\eta^2(q^2) = \sum_{\substack{k=-\infty \\ n\geq 0}}^{\infty} (-1)^{k+n}(2n+1) q^{\frac{(2k)^2+(2n+1)^2}{4}},$$

imply an eta product is lacunary, because subsequences of integers generated by quadratic forms have zero density. Unfortunately, it is not clear if every lacunary value of $e_a e_b e_c e_d$ possesses such an expansion. There is also insufficient evidence to conjecture how often $e_a e_b e_c e_d$ is lacunary. This stems from the fact that it is difficult to detect the property numerically. It requires thousands of q-series coefficients to convincingly demonstrate that (easy) cases like e_1^4 are lacunary. The calculations become much worse for more complicated examples.

The lattice sums $F(1,1)$, $F(1,2)$, and $F(1,3)$ equal L-values of CM elliptic curves. Therefore they are lacunary. These examples, and additional values such as $F(1,4)$ and $F(2,2)$, reduce to two-dimensional sums via classical theta series results. Less obvious lacunary sums include $F(2,9)$ and $F(4,7,7,28)$. These cases require eta function product identities. A result of Ramanujan [3, p. 210, Entry 56], shows that

$$3e_1 e_2 e_9 e_{18} = -e_1^2 e_2^2 + e_1^3 \frac{e_{18}^2}{e_9} + e_2^3 \frac{e_9^2}{e_{18}}. \qquad (16)$$

Substituting classical theta expansions for e_1^3, e_2^2/e_1, and e_1^2/e_2 [11, pg. 114-117], leads to

$$\begin{aligned} 3\eta(q)\eta(q^2)\eta(q^9)\eta(q^{18}) = &- \sum_{\substack{n=0\\k=0}}^{\infty} (-1)^n (2n+1) q^{\frac{(2n+1)^2+(2k+1)^2}{8}} \\ &+ \sum_{\substack{n=0\\k=0}}^{\infty} (-1)^n (2n+1) q^{\frac{(2n+1)^2+9(2k+1)^2}{8}} \\ &+ \sum_{\substack{n=0\\k=-\infty}}^{\infty} (-1)^{n+k} (2n+1) q^{\frac{(2n+1)^2+9(2k)^2}{4}}. \end{aligned} \qquad (17)$$

The eta product equals a finite linear combination of two-dimensional theta functions. Therefore it is lacunary. Formula (17) is the main ingredient needed to relate $F(2,9)$ to hypergeometric functions and Mahler measures.

Theorem 2. *Let $t = \sqrt[4]{12}$, then the following identity is true:*

$$\frac{144}{25\pi^2}F(2,9) = -3m(4i) + 2m\left(\frac{1}{\sqrt{2}}\left(4-2t-2t^2+t^3\right)\right) + m\left(4i\left(7+4t+2t^2+t^3\right)\right). \tag{18}$$

Proof. The most difficult portion of the calculation is to find a two-dimensional theta series for $e_1e_2e_9e_{18}$. This task has been accomplished via an eta function product identity. The remaining calculations parallel those carried out in [17]. Integrating (17) leads to

$$\frac{3}{25}F(2,9) + F(1,2) = 4\sum_{\substack{n=0\\k=0}}^{\infty}\frac{(-1)^n(2n+1)}{\left((2n+1)^2+9(2k+1)^2\right)^2} + \sum_{\substack{n=0\\k=-\infty}}^{\infty}\frac{(-1)^{n+k}(2n+1)}{\left((2n+1)^2+9(2k)^2\right)^2}. \tag{19}$$

There are two possible formulas for $F(1,2)$ [16]:

$$F(1,2) = \frac{\pi^2}{8}m\left(2\sqrt{2}\right) = \frac{\pi^2}{16}m(4i). \tag{20}$$

By the formula for $F_{(1,2)}(3)$ in [17, Eq. 115], we also have

$$\sum_{\substack{n=0\\k=-\infty}}^{\infty}\frac{(-1)^{n+k}(2n+1)}{\left((2n+1)^2+9(2k)^2\right)^2} = \frac{\pi^2}{48}m\left(4i\left(7+4t+2t^2+t^3\right)\right), \tag{21}$$

where $t=\sqrt[4]{12}$. Next we evaluate the remaining term in (19). Notice that for $x>0$

$$\sum_{\substack{n=0\\k=0}}^{\infty}\frac{(-1)^n(2n+1)}{\left((2n+1)^2+x(2k+1)^2\right)^2} = \frac{\pi^2}{16}\int_0^\infty u\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\pi(n+1/2)^2u}\right)\left(\sum_{k=0}^{\infty}e^{-\pi x(k+1/2)^2u}\right)\mathrm{d}u.$$

By the involution for the weight-$3/2$ theta function

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\pi(n+1/2)^2u} = \frac{1}{u^{3/2}}\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\pi(n+1/2)^2\frac{1}{u}},$$

this becomes

$$\begin{aligned}
&\sum_{\substack{n=0\\k=0}}^{\infty}\frac{(-1)^n(2n+1)}{\left((2n+1)^2+x(2k+1)^2\right)^2}\\
&\quad=\frac{\pi^2}{16}\sum_{\substack{n=0\\k=0}}^{\infty}(-1)^n(2n+1)\int_0^{\infty}u^{-1/2}e^{-\pi\left((n+1/2)^2\frac{1}{u}+x(k+1/2)^2u\right)}\mathrm{d}u\\
&\quad=\frac{\pi^2}{16\sqrt{x}}\sum_{\substack{n=0\\k=0}}^{\infty}(-1)^n\frac{(2n+1)}{(2k+1)}e^{-\frac{\pi\sqrt{x}}{2}(2n+1)(2k+1)}\\
&\quad=\frac{\pi^2}{16\sqrt{x}}\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(\frac{1+e^{-\pi\sqrt{x}(n+1/2)}}{1-e^{-\pi\sqrt{x}(n+1/2)}}\right).
\end{aligned}$$

Applying formulas (1.6), (1.7), and (2.9) in [13], we have

$$\begin{aligned}
&=\frac{\pi^2}{32\sqrt{x}}\left(m\left(\frac{4}{\sqrt{\alpha_{x/4}}}\right)-m\left(\frac{4i\sqrt{1-\alpha_{x/4}}}{\sqrt{\alpha_{x/4}}}\right)\right)\\
&=\frac{\pi^2}{32\sqrt{x}}m\left(4\left(\frac{1-\sqrt{1-\alpha_{x/4}}}{1+\sqrt{1-\alpha_{x/4}}}\right)\right),
\end{aligned}$$

where α_x is the singular modulus (recall that $\alpha_x=1-\varphi^4(-e^{-\pi\sqrt{x}})/\varphi^4(e^{-\pi\sqrt{x}})$). The second degree modular equation shows that

$$\frac{1-\sqrt{1-\alpha_{x/4}}}{1+\sqrt{1-\alpha_{x/4}}}=\sqrt{\alpha_x},$$

and hence we obtain

$$\sum_{\substack{n=0\\k=0}}^{\infty}\frac{(-1)^n(2n+1)}{\left((2n+1)^2+x(2k+1)^2\right)^2}=\frac{\pi^2}{32\sqrt{x}}m\left(4\sqrt{\alpha_x}\right). \tag{22}$$

It is well known that α_n can be expressed in terms of class invariants if $n\in\mathbb{Z}$:

$$\alpha_n=\frac{1}{2}\left(1-\sqrt{1-1/G_n^{24}}\right).$$

The values of G_n have been extensively tabulated [4, p. 188]. Setting $n=9$ yields

$$\begin{aligned}
\alpha_9=&\frac{1}{2}\left(1-\sqrt{1-\left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^8}\right)\\
=&\frac{1}{2}\left(1-4t+t^3\right)\\
=&\frac{\left(4-2t-2t^2+t^3\right)^2}{32},
\end{aligned}$$

where $t = \sqrt[4]{12}$. Formula (22) reduces to

$$\sum_{\substack{n=0\\k=0}}^{\infty} \frac{(-1)^n(2n+1)}{\left((2n+1)^2+9(2k+1)^2\right)^2} = \frac{\pi^2}{96} m\left(\frac{1}{\sqrt{2}}\left(4-2t-2t^2+t^3\right)\right). \tag{23}$$

The proof of (18) follows from combining (19), (20), (21), and (23).

We have chosen to exclude the explicit formula for $F(4,7,7,28)$ from this paper[1]. It suffices to say that the sum reduces to an extremely unpleasant expression involving hypergeometric functions and Meijer G-functions. The key modular equation is due to Somos [21, Entry $q_{28,9,35}$]:

$$28e_4e_7^2e_{28} = -7e_1e_7^3 - \frac{e_1^5}{e_2^2}\frac{e_{14}^2}{e_7} + 8\frac{e_2^5}{e_1^2}e_{14}. \tag{24}$$

By classical theta expansions [11, pg. 114-117], the eta product becomes

$$\begin{aligned}28\eta(q^4)\eta^2(q^7)\eta(q^{28}) = &-7\sum_{\substack{n=-\infty\\k=0}}^{\infty}(-1)^{n+k}(2k+1)q^{\frac{(6n+1)^2+21(2k+1)^2}{24}}\\ &-\sum_{\substack{n=-\infty\\k=0}}^{\infty}(6n+1)q^{\frac{(6n+1)^2+21(2k+1)^2}{24}}\\ &+8\sum_{n,k=-\infty}^{\infty}(-1)^{n+k}(3n+1)q^{\frac{4(3n+1)^2+7(6k+1)^2}{12}}.\end{aligned}$$

As a result it is easy to see that $e_4e_7^2e_{28}$ is lacunary.

We believe that there are additional lacunary values of $F(a,b,c,d)$. It might be interesting to try to detect them numerically. Another possible extension of this research involves looking at linear combinations of lattice sums. One can prove that certain linear combinations of lattice sums reduce to Mahler measures. As an example, briefly consider the following modular equation [21, Entry $x_{50,6,81}$]:

$$5e_1e_2e_{25}e_{50} + 2e_1^2e_2e_{50} + 2e_1e_2^2e_{25} = -e_1^2e_2^2 + e_1^3\frac{e_{50}^2}{e_{25}} + e_2^3\frac{e_{25}^2}{e_{50}}. \tag{25}$$

All three eta quotients on the right-hand side of (25) have two-dimensional theta series expansions. As a result we can prove that

$$\begin{aligned}&\frac{5}{13^2}F(2,25) + \frac{2}{9^2}F(1,1,2,50) + \frac{2}{5^2}F(1,2,2,25)\\ &\quad = \frac{\pi^2}{80}\left(-5m(4i) + 2m(4\sqrt{\alpha_{25}}) + m\left(4i\sqrt{\frac{1-\alpha_{25}}{\alpha_{25}}}\right)\right),\end{aligned} \tag{26}$$

[1] The formula is available upon request

where $\alpha_{25}=\frac{1}{2^{13}}\left(\sqrt{5}-1\right)^8\left(\sqrt[4]{5}-1\right)^8$. There are many additional results like (26), which we will not discuss here.

3.2 Non-lacunary cases

The calculations become far more difficult when $F(a,b,c,d)$ *does not* reduce to a two-dimensional sum. The recent proofs of formulas for $F(1,5)$, $F(2,3)$, and $F(3,5)$, are all based upon new types of q-integral transformations [18], [19]. The fundamental transformation for $F(2,3)$ is

$$\int_0^1 q^{1/2}\psi(q)\psi(q^3)\varphi(-q^x)\varphi(-q^{3x})\log q\frac{\mathrm{d}q}{q}$$
$$=\frac{2\pi}{3x}\operatorname{Im}\int_0^1 \omega q\psi^4\left(\omega^2q^2\right)\log\left(4q^{3x}\frac{\psi^4(q^{12x})}{\psi^4(q^{6x})}\right)\frac{\mathrm{d}q}{q},$$

where $\omega=e^{2\pi i/3}$. When $x=1$ the left-hand side equals $-4F(2,3)$ (to see this use $q^{1/8}\psi(q)=\eta^2(q^2)/\eta(q)$ and $\varphi(-q)=\eta^2(q)/\eta(q^2)$), and the right-hand side becomes an extremely complicated elementary integral. The most difficult portion of the calculation is to reduce the elementary integral to hypergeometric functions,

$$F(2,3)=\frac{\pi^2}{6}m(2)=\frac{\pi^2}{12}{}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop 1,\frac{3}{2}};\frac{1}{4}\right).$$

Boyd's numerical work was instrumental in the calculation, because it allowed the final formula to be anticipated in advance.

Non-lacunary lattice sums reduce to intractable integrals quite frequently. We recently used the method from [18] to find a formula for $F(1,8)$:

$$F(1,8)=\frac{9\pi\sqrt[4]{2}}{128}\int_0^1\frac{(1-k)^2+2\sqrt{2(k+k^3)}}{(1+k)(k+k^3)^{3/4}}\log\left(\frac{1+2k-k^2+2\sqrt{k-k^3}}{1+k^2}\right)\mathrm{d}k \tag{27}$$

The proof of (27) is long and complicated, so we verified this monstrous identity to 100 decimal places by calculating $F(1,8)$ with (12). We speculate that the integral should reduce to something along the lines of (18).

Eta function identities occasionally provide shortcuts for avoiding integrals like (27). We have already demonstrated that linear dependencies exist between lattice sums (see (26)). In certain cases it is possible to relate new lattice sums to well-known examples. Consider a forty-fifth degree modular equation due to Somos [21, Entry $x_{45,4,12}$]:

$$6e_1e_5e_9e_{45}=-e_1^2e_5^2-2e_3^2e_{15}^2-9e_9^2e_{45}^2+e_3^4+5e_{15}^4. \tag{28}$$

We were unable to prove (28) by elementary methods. Integrating (28) leads to a linear dependency between three lattice sums. We have

$$9F(5,9) = 45F(1,1) - 50F(1,5). \tag{29}$$

Both $F(1,1)$ and $F(1,5)$ equal values of hypergeometric functions [16], [18], so we easily obtain the following theorem.

Theorem 3. *Recall that $n(\alpha)$ is defined in* (2)*. We have*

$$\frac{108}{5\pi^2}F(5,9) = 8n\left(3\sqrt[3]{2}\right) - 9n\left(2\sqrt[3]{4}\right). \tag{30}$$

Boyd's Mahler measure conjectures imply various additional formulas. A proof of Boyd's conductor 30 conjectures would lead to closed forms for both $F(2,15)$ and $F(2,5/3)$. To make this explicit we use two relations. First consider a four term modular equation due to Somos [20]:

$$e_1e_3e_5e_{15} + 2e_2e_6e_{10}e_{30} = e_1e_2e_{15}e_{30} + e_3e_5e_6e_{10}. \tag{31}$$

Integrating (31), and then using the evaluation $F(3,5) = 4\pi^2 m(1)/15$ from [19], leads to

$$F(2,15) + 4F\left(2,\frac{5}{3}\right) = \frac{8\pi^2}{5}m(1). \tag{32}$$

Next we require an unproven relation. Boyd conjectured[2] that for a conductor 30 elliptic curve

$$L(E_{30},2) \stackrel{?}{=} \frac{2\pi^2}{15}g(3),$$

where $g(\alpha)$ is defined in (3). The modularity theorem guarantees that $L(E_{30},2) = L(f_{30},2)$, where $f_{30}(e^{2\pi i\tau})$ is a weight-two cusp form on $\Gamma_0(30)$. Somos has calculated a basis for the space of cusp forms on $\Gamma_0(30)$. It follows that the cusp form associated with conductor 30 elliptic curves is

$$f_{30}(q) = \eta(q^3)\eta(q^5)\eta(q^6)\eta(q^{10}) - \eta(q)\eta(q^2)\eta(q^{15})\eta(q^{30}).$$

Upon integrating $f_{30}(q)$, Boyd's conjecture becomes

$$F\left(2,\frac{5}{3}\right) - \frac{1}{4}F(2,15) \stackrel{?}{=} \frac{2\pi^2}{15}g(3). \tag{33}$$

Combining (32) and (33) leads to a pair of conjectural evaluations.

Conjecture 1. Recall that $m(\alpha)$ and $g(\alpha)$ are defined in (1) and (3). The following equivalent formulas are numerically true:

[2] See Table 2 in [6]. In our notation, Boyd's entries correspond to values of $g(2-k)$

$$\frac{15}{4\pi^2}F(2,15) \stackrel{?}{=} 3m(1) - g(3), \tag{34}$$

$$\frac{15}{\pi^2}F\left(2,\frac{5}{3}\right) \stackrel{?}{=} 3m(1) + g(3). \tag{35}$$

Tracking backwards shows that a proof of either (34) or (35) would settle Boyd's conductor 30 Mahler measure conjectures. Proofs remain out of reach, however we are optimistic that both identities may be proved using Eisenstein series identities due to Berkovich and Yesilyurt [1].

4 Conclusion: Conductor 17 elliptic curves

An important connection exists between lattice sums and Mahler measures, however this relationship has limitations. Even if we could "sum up" $F(b,c)$ for arbitrary values of b and c, this would only settle a few of Boyd's conjectures [6]. Conductor 17 curves are the first cases in Cremona's list [8], where $L(E,2)$ probably does not reduce to values of $F(b,c)$. If we let E_{17} denote a conductor 17 curve (we used $y^2+xy+y=x^3-x^2-x$), then

$$\frac{17}{2\pi^2}L(E_{17},2) \stackrel{?}{=} m\left(\frac{(1+\sqrt{17})^2}{4}\right) - m\left(\sqrt{17}\right). \tag{36}$$

We discovered (36) via numerical experiments involving elliptic dilogarithms. [3] The cusp form associated with conductor 17 curves is stated in [10]. We have

$$f_{17}(q) = \frac{\eta(q)\eta^2(q^4)\eta^5(q^{34})}{\eta(q^2)\eta(q^{17})\eta^2(q^{68})} - \frac{\eta^5(q^2)\eta(q^{17})\eta^2(q^{68})}{\eta(q)\eta^2(q^4)\eta(q^{34})}. \tag{37}$$

Since $L(E_{17},2) = L(f_{17},2)$, formula (36) can be changed into a complicated elementary identity. There does not seem to be an easy way to relate $L(E_{17},2)$ to Mahler measures of rational polynomials. This probably explains why conductor 17 curves never appear in Boyd's paper [6]. After examining $f_{17}(q)$ in detail, we feel reasonably confident that $L(E_{17},2)$ is linearly independent from values of $F(b,c)$ over $\mathbb{Q}$.

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[3] Brunault recently informed us that he can prove (36) with a method based upon Beilinson's theorem [7].

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