New ${}_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$

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Abstract

New relations are established between families of three-variable Mahler measures. Those identities are then expressed as transformations for the ${}_5F_4$ hypergeometric function. We use these results to obtain two explicit ${}_5F_4$ evaluations, and several new formulas for $1/\pi$.

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1 Introduction

In this paper we will study the consequences of some recent results of Bertin. Recall that Bertin proved q-series expansions for a pair of three-variable Mahler measures in [8]. As usual the Mahler measure of an n-variable polynomial, $P(z_1,\dots,z_n)$, is defined by

$$\mathrm{m}\left(P(z_1,\dots,z_n)\right)=\int_0^1\dots\int_0^1\log\left|P\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\dots,\mathrm{e}^{2\pi\mathrm{i}\theta_n}\right)\right|\mathrm{d}\theta_1\dots\mathrm{d}\theta_n.$$

We will define $g_1(u)$ and $g_2(u)$ in terms of the following three-variable Mahler measures

$$g_1(u):=\mathrm{m}\left(u+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}\right),\tag{1.1}$$

$$\begin{aligned}g_2(u):=\mathrm{m}&\left(-u+4+\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right.\\&\left.+\left(x+x^{-1}\right)\left(z+z^{-1}\right)+\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right).\end{aligned}\tag{1.2}$$

We can recover Bertin's original notation by observing that $g_1(u) = \mathrm{m}(P_u)$, and after substituting $(xz, y/z, z/x) \to (x, y, z)$ in Eq. (1.2) we see that $g_2(u+4) = \mathrm{m}(Q_u)$ [8].

In Section 2 we will show how to establish a large number of interesting relations between $g_1(u)$, $g_2(u)$, and three more three-variable Mahler measures. For example, for $|u|$ sufficiently large Eq. (2.21) is equivalent to

$$\begin{aligned} g_1\left(3\left(u^2+u^{-2}\right)\right) =& \frac{1}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{(3+u^4)}{u^3}xyz\right) \\ &+\frac{3}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{(3+u^{-4})}{u^{-3}}xyz\right). \end{aligned} \tag{1.3}$$

Rodriguez-Villegas briefly mentioned the Mahler measure $\mathrm{m}\left(x^4+y^4+z^4+1+uxyz\right)$ on the last page of [17].

We will also show that identities like Eq. (1.3) are equivalent to transformations for the ${}_5F_4$ hypergeometric function. Recall that the generalized hypergeometric function is defined by

$${}_pF_q\left({a_1,a_2,\ldots a_p \atop b_1,b_2,\ldots b_q};x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{n!},$$

where $(y)_n = \Gamma(y+n)/\Gamma(y)$. We have restated Eq. (1.3) as a hypergeometric transformation in Eq. (2.24). As a special case of Eq. (1.3) we can also deduce that

$${}_5F_4\left({\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1 \atop 2,2,2,2};1\right) = \frac{256}{3}\log(2) - \frac{5120\sqrt{2}}{3\pi^3}L(f,3),$$

where $f(q) = q\prod_{n=1}^{\infty}(1-q^n)^2\left(1-q^{2n}\right)\left(1-q^{4n}\right)\left(1-q^{8n}\right)^2$, and $L(f,s)$ is the usual L-series of $f(q)$. We will conclude Section 2 with a brief discussion of some related, but still unproven, evaluations of the ${}_4F_3$ and ${}_3F_2$ hypergeometric functions.

It turns out that $g_1(u)$ and $g_2(u)$ are also closely related to Watson's triple integrals. For appropriate values of u, Watson showed that $g_1'(u)$ and $g_2'(u)$ reduce to products of elliptical integrals (for relevant results see [18], [13], [14], and [12]). In Section 3 we will use some related transformations to prove new formulas for $1/\pi$. For example, we will show that

$$\frac{2}{\pi} = \sum_{n=0}^{\infty}(-1)^n\frac{(3n+1)}{32^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2.$$

Notice that this formula for $1/\pi$ involves the Domb numbers. Chan, Chan and Liu obtained a similar formula for $1/\pi$ involving Domb numbers in [10], we have recovered their result in (3.10). Zudilin and Yang also discovered some related formulas for $1/\pi$ in [20]. All of the ${}_3F_2$ transformations that we will utilize in Section 3 follow from differentiating the ${}_5F_4$ identities established in Section 2.

2 Identities between Mahler measures and transformations for the ${}_5F_4$ function

Bertin proved that both $g_1(u)$ and $g_2(u)$ have convenient q-series expansions when u is parameterized correctly. Before stating her theorem, we will define some notation. As usual let

$$(x,q)_\infty = (1-x)(1-xq)\left(1-xq^2\right)\dots,$$

and define $G(q)$ by

$$G(q) = \mathrm{Re}\left[-\log(q) + 240\sum_{n=1}^{\infty} n^2 \log\left(1-q^n\right)\right]. \tag{2.1}$$

Notice that if $q \in (0,1)$ then $G'(q) = -M(q)/q$, where $M(q)$ is the Eisenstein series of weight 4 on the full modular group $\Gamma(1)$ [3].

Theorem 2.1. *(Bertin) For $|q|$ sufficiently small*

$$g_1\left(t_1(q)\right) = -\frac{1}{60}G(q) + \frac{1}{30}G(q^2) - \frac{1}{20}G(q^3) + \frac{1}{10}G(q^6), \tag{2.2}$$

$$g_2\left(t_2(q)\right) = \frac{1}{120}G(q) - \frac{1}{15}G\left(q^2\right) - \frac{1}{40}G\left(q^3\right) + \frac{1}{5}G\left(q^6\right), \tag{2.3}$$

where

$$t_1(q) = v_1 + \frac{1}{v_1}, \text{ and } v_1 = q^{1/2}\frac{\left(q;q^2\right)_\infty^6}{\left(q^3;q^6\right)_\infty^6},$$

$$t_2(q) = -\left(v_2 - \frac{1}{v_2}\right)^2, \text{ and } v_2 = q^{1/2}\frac{\left(q^2;q^2\right)_\infty^6\left(q^3;q^3\right)_\infty^2\left(q^{12};q^{12}\right)_\infty^4}{\left(q;q\right)_\infty^2\left(q^4;q^4\right)_\infty^4\left(q^6;q^6\right)_\infty^6}.$$

In this section we will show that both $g_1(u)$ and $g_2(u)$ reduce to linear combinations of ${}_5F_4$ hypergeometric functions. We will accomplish this goal by first expressing each of the functions

$$f_2(u) := 2\mathrm{m}\left(u^{1/2} + \left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)\right), \tag{2.4}$$

$$f_3(u) := \mathrm{m}\left(u - \left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2(1+z)^3 z^{-2}\right), \tag{2.5}$$

$$f_4(u) := 4\mathrm{m}\left(x^4+y^4+z^4+1+u^{1/4}xyz\right), \tag{2.6}$$

in terms of $G(q)$. We will then exploit those identities to establish linear relations between functions in the set $\{f_2(u), f_3(u), f_4(u), g_1(u), g_2(u)\}$. This is significant since $f_2(u)$, $f_3(u)$, and $f_4(u)$ all reduce to ${}_5F_4$ hypergeometric functions. In particular this implies the non-trivial fact that both $g_1(u)$ and $g_2(u)$ also reduce to linear combinations of ${}_5F_4$ functions.

Proposition 2.2. *The following identities hold for $|u|$ sufficiently large:*

$$f_2(u) = Re\left[\log(u) - \frac{8}{u}{}_5F_4\left(\begin{smallmatrix}\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1\\ 2,2,2,2\end{smallmatrix};\frac{64}{u}\right)\right], \tag{2.7}$$

$$f_3(u) = Re\left[\log(u) - \frac{12}{u}{}_5F_4\left(\begin{smallmatrix}\frac{4}{3},\frac{3}{2},\frac{5}{3},1,1\\ 2,2,2,2\end{smallmatrix};\frac{108}{u}\right)\right], \tag{2.8}$$

$$f_4(u) = Re\left[\log(u) - \frac{24}{u}{}_5F_4\left(\begin{smallmatrix}\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1\\ 2,2,2,2\end{smallmatrix};\frac{256}{u}\right)\right]. \tag{2.9}$$

For $|u| > 6$

$$g_1(u) = Re\left[\log(u) - \sum_{n=1}^{\infty}\frac{(1/u)^{2n}}{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\right], \tag{2.10}$$

and if $|u| > 16$

$$g_2(u) = Re\left[\log(u) - \sum_{n=1}^{\infty}\frac{(1/u)^{n}}{n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2\right]. \tag{2.11}$$

Proof. We can prove each of these identities using a method due to Rodriguez-Villegas [17]. We will illustrate the proof of Eq. (2.7) explicitly. Rearranging the Mahler measure shows that

$$f_2(u) = \mathrm{Re}\left[\log(u) + \int_0^1\int_0^1\int_0^1 \log\left(1 - \frac{64}{u}\cos^2(2\pi t_1)\cos^2(2\pi t_2)\cos^2(2\pi t_3)\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\right].$$

If $|u| > 64$, then $\left|\frac{64}{u}\cos^2(2\pi t_1)\cos^2(2\pi t_2)\cos^2(2\pi t_3)\right| < 1$, hence by the Taylor series for the logarithm

$$\begin{aligned} f_2(u) =& \mathrm{Re}\left[\log(u) - \sum_{n=1}^{\infty}\frac{(64/u)^n}{n}\left(\int_0^1\cos^{2n}(2\pi t)\mathrm{d}t\right)^3\right] \\ =& \mathrm{Re}\left[\log(u) - \sum_{n=1}^{\infty}\binom{2n}{n}^3\frac{(1/u)^n}{n}\right] \\ =& \mathrm{Re}\left[\log(u) - \frac{8}{u}{}_5F_4\left(\begin{smallmatrix}\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1\\ 2,2,2,2\end{smallmatrix};\frac{64}{u}\right)\right]. \end{aligned}$$

Notice that Eq. (2.7) holds whenever $u \notin [-64, 64]$, since $f_2(u)$ is harmonic in $\mathbb{C}\backslash[-64, 64]$.■

While Proposition 2.2 shows that the results in this paper easily translate into the language of hypergeometric functions, the relationship to Mahler measure is more important than simple pedagogy. Bertin proved that for certain values of u the zero varieties of the (projectivized) polynomials from equations (1.1) and (1.2) define $K3$ hypersurfaces. She also proved formulas relating the L-functions of these

$K3$ surfaces at $s = 3$ to rational multiples of the Mahler measures. Proposition 2.2 shows that these results imply explicit ${}_5F_4$ evaluations (see Corollary 2.6 for explicit examples). While it might also be interesting to interpret the polynomials from equations (2.4) through (2.6) in terms of $K3$ hypersurfaces, we will not pursue that direction here.

Theorem 2.3. *For $|q|$ sufficiently small*

$$f_2(s_2(q)) = -\frac{2}{15}G(q) - \frac{1}{15}G(-q) + \frac{3}{5}G\left(q^2\right), \tag{2.12}$$
$$f_3(s_3(q)) = -\frac{1}{8}G(q) + \frac{3}{8}G\left(q^3\right), \tag{2.13}$$
$$f_4(s_4(q)) = -\frac{1}{3}G(q) + \frac{2}{3}G\left(q^2\right), \tag{2.14}$$

where

$$\begin{aligned} s_2(q) =& q^{-1}\left(-q;q^2\right)_\infty^{24}, \\ s_3(q) =& \frac{1}{q}\left(27q\frac{\left(q^3;q^3\right)_\infty^6}{(q;q)_\infty^6} + \frac{(q;q)_\infty^6}{\left(q^3;q^3\right)_\infty^6}\right)^2, \\ s_4(q) =& \frac{1}{q}\frac{\left(q^2;q^2\right)_\infty^{24}}{(q;q)_\infty^{24}}\left(16q\frac{(q;q)_\infty^4\left(q^4;q^4\right)_\infty^8}{\left(q^2;q^2\right)_\infty^{12}} + \frac{\left(q^2;q^2\right)_\infty^{12}}{(q;q)_\infty^4\left(q^4;q^4\right)_\infty^8}\right)^4. \end{aligned}$$

The following inverse relations hold for $|q|$ sufficiently small:

$$G(q) = -19f_2\left(s_2(q)\right) - 4f_2\left(s_2(-q)\right) + 24f_2\left(s_2\left(q^2\right)\right) - 12f_2\left(s_2\left(-q^2\right)\right), \tag{2.15}$$
$$\begin{aligned} G(q) = &-\frac{19}{2}f_3\left(s_3(q)\right) - \frac{3}{2}f_3\left(s_3\left(e^{2\pi i/3}q\right)\right) \\ &-\frac{3}{2}f_3\left(s_3\left(e^{4\pi i/3}q\right)\right) + \frac{9}{2}f_3\left(s_3\left(q^3\right)\right), \end{aligned} \tag{2.16}$$
$$G(q) = -5f_4\left(s_4(q)\right) - 2f_4\left(s_4(-q)\right) + 4f_4\left(s_4\left(q^2\right)\right). \tag{2.17}$$

Proof. We can use Ramanujan's theory of elliptic functions to verify the first half of this theorem. Recall that the *elliptic nome* is defined by

$$q_j(\alpha) = \exp\left(-\frac{\pi}{\sin(\pi/j)}\frac{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; \alpha\right)}\right).$$

It is a well established fact that $s_j\left(q_j\left(\alpha\right)\right)$ is a rational function of α whenever $j \in \{2,3,4\}$. For example if $q = q_2(\alpha)$, then $s_2\left(q\right) = \frac{16}{\alpha(1-\alpha)}$. Therefore we can verify Eq. (2.12) by differentiating with respect to α, and by showing that the identity holds when $q \to 0$.

Observe that when $q \to 0$ both sides of Eq. (2.12) approach $-\log|q| + O(q)$. Differentiating with respect to α yields

$$-\frac{(1-2\alpha)}{2\alpha(1-\alpha)}{}_3F_2\left({\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right)$$
$$= -\frac{1}{2q}\left(1 - 16\sum_{n=1}^{\infty} n^3\frac{q^n}{1-q^n} + 256\sum_{n=1}^{\infty} n^3\frac{q^{4n}}{1-q^{4n}}\right)\frac{\mathrm{d}q}{\mathrm{d}\alpha}.$$

This final identity follows from applying three well known formulas:

$$\frac{\mathrm{d}q}{\mathrm{d}\alpha} = \frac{q}{\alpha(1-\alpha){}_2F_1{}^2\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)},$$
$${}_3F_2\left({\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2} \atop 1,1};4\alpha(1-\alpha)\right) = {}_2F_1{}^2\left(\frac{1}{2},\frac{1}{2};1;\alpha\right),$$
$$1 - 16\sum_{n=1}^{\infty} n^3\frac{q^n}{1-q^n} + 256\sum_{n=1}^{\infty} n^3\frac{q^{4n}}{1-q^{4n}} = (1-2\alpha){}_2F_1{}^4\left(\frac{1}{2},\frac{1}{2};1;\alpha\right).$$

We can verify equations (2.13) and (2.14) in a similar manner by using the fact that $s_3\left(q_3(\alpha)\right) = \frac{27}{\alpha(1-\alpha)}$, and $s_4\left(q_4(\alpha)\right) = \frac{64}{\alpha(1-\alpha)}$.

The crucial observation for proving equations (2.15) through (2.17) is the fact that $G(q)$ satisfies the following functional equation for any prime p:

$$\sum_{j=0}^{p-1} G\left(\mathrm{e}^{2\pi \mathrm{i} j/p} q\right) = \left(1+p^3\right)G\left(q^p\right) - p^2 G\left(q^{p^2}\right). \tag{2.18}$$

We will only need the $p=2$ case to prove Eq. (2.15):

$$G(q) + G(-q) = 9G\left(q^2\right) - 4G\left(q^4\right).$$

Notice that this last formula always allows us to eliminate $G\left(q^4\right)$ from an equation. Applying the substitutions $q \to -q$, $q \to q^2$, and $q \to -q^2$ to Eq. (2.12) yields

$$\begin{pmatrix} -2/15 & -1/15 & 3/5 \\ -1/15 & -2/15 & 3/5 \\ -3/20 & -3/20 & 23/20 \end{pmatrix}\begin{pmatrix} G(q) \\ G(-q) \\ G\left(q^2\right)\end{pmatrix} = \begin{pmatrix} f_2\left(s_2(q)\right) \\ f_2\left(s_2(-q)\right) \\ 2f_2\left(s_2\left(q^2\right)\right) - f_2\left(s_2\left(-q^2\right)\right)\end{pmatrix}.$$

Since this system of equations is non-singular, we can invert the matrix to recover Eq. (2.15). We can prove equations (2.16) and (2.17) in a similar fashion.■

If we compare Theorem 2.3 with Bertin's results we can deduce some obvious relationships between the Mahler measures. For example, combining Eq. (2.14) with Eq. (2.3), and combining Eq. (2.13) with Eq. (2.2), we find that

$$g_1(t_1(q)) = \frac{1}{20}f_4\left(s_4(q)\right) + \frac{3}{20}f_4\left(s_4\left(q^3\right)\right), \tag{2.19}$$
$$g_2\left(t_2(q)\right) = -\frac{1}{15}f_3\left(s_3(q)\right) + \frac{8}{15}f_3\left(s_3\left(q^2\right)\right). \tag{2.20}$$

Notice that many more identities follow from substituting equations (2.15) through (2.17) into formulas (2.2), (2.3), (2.12), (2.13) and (2.14). However, for the remainder of this section we will restrict our attention to equations (2.19) and (2.20). In particular, we will appeal to the theory of elliptic functions to transform those results into identities which depend on rational arguments.

If we let $q = q_2(\alpha)$, then it is well known that $\frac{q^{j/24}(q^j;q^j)_\infty}{q^{1/24}(q;q)_\infty}$ is an algebraic function of α for $j \in \{1,2,3,\dots\}$ (for example see [4] or [5]). It follows immediately that $s_2(q)$, $s_3(q)$, $s_4(q)$, $t_1(q)$, and $t_2(q)$ are also algebraic functions of α. The following lemma lists several instances where those functions have rational parameterizations.

Lemma 2.4. *Suppose that $q = q_2(\alpha)$, where $\alpha = p(2+p)^3/(1+2p)^3$. The following identities hold for $|p|$ sufficiently small:*

$$s_2(q) = \frac{16(1+2p)^6}{p(1-p)^3(1+p)(2+p)^3}, \qquad s_2\left(q^3\right) = \frac{16(1+2p)^2}{p^3(1-p)(1+p)^3(2+p)},$$

$$s_2(-q) = -\frac{16(1-p)^6(1+p)^2}{p(2+p)^3(1+2p)^3}, \qquad s_2\left(-q^3\right) = -\frac{16(1-p)^2(1+p)^6}{p^3(2+p)(1+2p)},$$

$$s_2\left(-q^2\right) = \frac{16^2(1-p)^3(1+p)(1+2p)^3}{p^2(2+p)^6}, \qquad s_2\left(-q^6\right) = \frac{16^2(1-p)(1+p)^3(1+2p)}{p^6(2+p)^2},$$

$$s_3(q) = \frac{4\left(1+4p+p^2\right)^6}{p\left(1-p^2\right)^4(2+p)(1+2p)}, \qquad s_3\left(q^2\right) = \frac{16\left(1+p+p^2\right)^6}{p^2\left(1-p^2\right)^2(2+p)^2(1+2p)^2},$$

$$s_3(-q) = -\frac{4\left(1-2p-2p^2\right)^6}{p(1-p^2)(2+p)(1+2p)^4}, \qquad s_3\left(q^4\right) = \frac{4\left(2+2p-p^2\right)^6}{p^4\left(1-p^2\right)(2+p)^4(1+2p)},$$

$$s_4(q) = \frac{16\left(1+14p+24p^2+14p^3+p^4\right)^4}{p(1-p)^6(1+p)^2(2+p)^3(1+2p)^3},$$

$$s_4\left(q^3\right) = \frac{16\left(1+2p+2p^3+p^4\right)^4}{p^3(1-p)^2(1+p)^6(2+p)(1+2p)},$$

$$s_4(-q) = -\frac{16\left(1-10p-12p^2-4p^3-2p^4\right)^4}{p(1-p)^3(1+p)(1+2p)^6(2+p)^3},$$

$$s_4\left(-q^3\right) = -\frac{16\left(1+2p-4p^3-2p^4\right)^4}{p^3(1-p)(1+p)^3(1+2p)^2(2+p)}.$$

Rational formulas also exist for certain values of $t_1^2(q)$ and $t_2(q)$:

$$t_1^2(q) = \frac{4\left(1+p+p^2\right)^2\left(1+4p+p^2\right)^2}{p(1-p^2)^2(2+p)(1+2p)},$$

$$t_1^2(-q) = -\frac{4\left(1+p+p^2\right)^2\left(1-2p-2p^2\right)^2}{p(1-p^2)(2+p)(1+2p)^2},$$

$$t_2(q) = -\frac{4\left(1-p^2\right)^2}{p(2+p)(1+2p)},$$

$$t_2(-q) = -\frac{4(1+p+p^2)^2}{p(1-p^2)(2+p)}.$$

The main difficulty with Lemma 2.4 is the fact that very few values of $s_j(\pm q^n)$ reduce to rational functions of p. Consider the set $\{s_2(q), s_2(-q), s_2\left(-q^2\right), s_2\left(q^2\right)\}$ as an example. While Lemma 2.4 shows that $s_2(q)$, $s_2(-q)$, and $s_2\left(-q^2\right)$ are all rational with respect to p, the formula for $s_2\left(q^2\right)$ involves radicals. Recall that if $\alpha = p(2+p)^3/(1+2p)^3$, then

$$s_2\left(q^2\right) = \frac{4\left(1+\sqrt{1-\alpha}\right)^6}{\alpha^2\sqrt{1-\alpha}},$$

where $\sqrt{1-\alpha} = \frac{1-p}{(1+2p)^2}\sqrt{(1-p^2)(1+2p)}$. Since the curve $X^2 = (1-p^2)(1+2p)$ is elliptic with conductor 24, it follows immediately that rational substitutions for p will never reduce $s_2\left(q^2\right)$ to a rational function. For the sake of legibility, we will therefore avoid all identities which involve those four functions simultaneously. By avoiding pitfalls of this nature, we can derive several interesting results from Lemma 2.4.

Theorem 2.5. *For $|z|$ sufficiently large*

$$g_1\left(3\left(z+z^{-1}\right)\right) = \frac{1}{20} f_4\left(\frac{9\left(3+z^2\right)^4}{z^6}\right) + \frac{3}{20} f_4\left(\frac{9\left(3+z^{-2}\right)^4}{z^{-6}}\right), \tag{2.21}$$

$$g_2(z) = -\frac{1}{15} f_3\left(\frac{(16-z)^3}{z^2}\right) + \frac{8}{15} f_3\left(-\frac{(4-z)^3}{z}\right). \tag{2.22}$$

Proof. These identities follow from applying Lemma 2.4 to equations (2.19) and (2.20). If we consider Eq. (2.19), then Lemma 2.4 shows that $t_1^2(q)$, $s_4(q)$, and $s_4\left(q^3\right)$ are all rational functions of p. Forming a resultant with respect to p, we obtain

$$0 = \operatorname*{Res}_p \left[\frac{4\left(1+p+p^2\right)^2\left(1+4p+p^2\right)^2}{p\left(1-p^2\right)^2(2+p)(1+2p)} - t_1^2(q), \right.$$
$$\left. \frac{16\left(1+14p+24p^2+14p^3+p^4\right)^4}{p(1-p)^6(1+p)^2(2+p)^3(1+2p)^3} - s_4(q)\right].$$

Simplifying with the aid of a computer, this becomes

$$0 = s_4^2(q) + \left(12+t_1^2(q)\right)^4 - s_4(q)\left(-288+352t_1^2(q) - 42t_1^4(q) + t_1^6(q)\right).$$

If we choose z so that $t_1(q) = 3\left(z+z^{-1}\right)$, then $s_4(q) = 9\left(3+z^2\right)^4 z^{-6}$, and a formula for $s_4\left(q^3\right)$ follows in a similar fashion.■

If we let $u = 1/z$ with $z \in \mathbb{R}$ and sufficiently large, then Eq. (2.22) reduces to the following infinite series identity:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{u^n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2 \\ =&\frac{1}{5}\log\left(\frac{(1-16u)}{(1-4u)^8}\right)+\frac{4u}{5(1-16u)^3}{}_5F_4\left({\tfrac{4}{3},\tfrac{3}{2},\tfrac{5}{3},1,1 \atop 2,2,2,2};-\frac{108u}{(1-16u)^3}\right) \\ &+\frac{32u^2}{5(1-4u)^3}{}_5F_4\left({\tfrac{4}{3},\tfrac{3}{2},\tfrac{5}{3},1,1 \atop 2,2,2,2};\frac{108u^2}{(1-4u)^3}\right).\end{aligned} \tag{2.23}$$

Similarly, if we let $u = 1/z^2$ then Eq. (2.21) is equivalent to:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{u}{9(1+u)^2}\right)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2 \\ =&\frac{2}{5}\log\left(\frac{27(1+u)^5}{(3+u)^3(1+3u)}\right)+\frac{4u^3}{5(3+u)^4}{}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};\frac{256u^3}{9(3+u)^4}\right) \\ &+\frac{4u}{15(1+3u)^4}{}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};\frac{256u}{9(1+3u)^4}\right).\end{aligned} \tag{2.24}$$

In Section 3 we will differentiate equations (2.23) and (2.24) to obtain several new formulas for $1/\pi$. But first we will conclude this section by deducing some explicit ${}_5F_4$ evaluations.

Recall that for certain values of u, Bertin evaluated $g_1(u)$ and $g_2(u)$ in terms of the L-series of $K3$ surfaces. She also proved equivalent formulas involving twisted cusp forms. Amazingly, her formulas correspond to cases where the right-hand sides of equations (2.21) and (2.22) collapse to one hypergeometric term. We can combine her results with equations (2.23) and (2.24) to deduce several new ${}_5F_4$ evaluations.

Corollary 2.6. *If* $g(q) = q\left(q^2;q^2\right)_\infty^3\left(q^6;q^6\right)_\infty^3$, *then*

$${}_5F_4\left({\tfrac{4}{3},\tfrac{3}{2},\tfrac{5}{3},1,1 \atop 2,2,2,2};1\right) = 18\log(2)+27\log(3)-\frac{810\sqrt{3}}{\pi^3}L(g,3). \tag{2.25}$$

If $f(q) = q\left(q;q\right)_\infty^2\left(q^2;q^2\right)_\infty\left(q^4;q^4\right)_\infty\left(q^8;q^8\right)_\infty^2$, *then*

$${}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};1\right) = \frac{256}{3}\log(2)-\frac{5120\sqrt{2}}{3\pi^3}L(f,3). \tag{2.26}$$

While many famous ${}_5F_4$ identities, such as Dougall's formula [3], reduce special values of the ${}_5F_4$ function to gamma functions, equations (2.25) and (2.26) do not fit into this category. Rather these new formulas are higher dimensional analogues of Boyd's conjectures. In particular, Boyd has conjectured large numbers of identities relating two-variable Mahler measures (that mostly reduce to ${}_4F_3$ functions) to the

L-series of elliptic curves [9]. The most famous outstanding conjecture of this type asserts that

$$\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) = -2\mathrm{Re}\left[{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};16\right)\right] \stackrel{?}{=} \frac{15}{4\pi^2}L(f,2),$$

where

$$f(q) = q\prod_{n=1}^{\infty}(1-q^n)\left(1-q^{3n}\right)\left(1-q^{5n}\right)\left(1-q^{15n}\right),$$

and "$\stackrel{?}{=}$" indicates numerical equality to at least 50 decimal places. Recently, Kurokawa and Ochiai proved a formula [15] which simplifies this last conjecture to

$${}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{3}{2},1};\frac{1}{16}\right) \stackrel{?}{=} \frac{15}{\pi^2}L(f,2).$$

Of course it would be highly desirable to rigorously prove Boyd's conjectures. Failing that, it might be interesting to search for more hypergeometric identities like equations (2.25) and (2.26). This line of thought suggests the following fundamental problem with which we shall conclude this section:

Open Problem: Determine every L-series that can be expressed in terms of generalized hypergeometric functions with algebraic parameters.

3 New formulas for $1/\pi$

In the previous section we produced several new transformations for the ${}_5F_4$ hypergeometric function. Now we will differentiate those formulas to obtain some new ${}_3F_2$ transformations, and several accompanying formulas for $1/\pi$. The following formula is a typical example of the identities in this section:

$$\frac{2}{\pi} = \sum_{n=0}^{\infty}(-1)^n\frac{(3n+1)}{32^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2. \tag{3.1}$$

Ramanujan first proved identities like Eq. (3.1) in his famous paper "Modular equations and approximations to π" [16]. He showed that the following infinite series holds for certain constants A, B, and X:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}(An+B)\frac{(1/2)_n^3}{n!^3}X^n \tag{3.2}$$

Ramanujan determined many sets of algebraic values for A, B, and X by expressing them in terms of the classical singular moduli G_n and g_n. He also stated (but did not prove) several formulas for $1/\pi$ where $(1/2)_n^3$ is replaced by $(1/a)_n\,(1/2)_n\,(1-1/a)_n$ for $a \in \{3,4,6\}$ (for more details see [6] or [11]).

Ramanujan's formulas for $1/\pi$ have attracted a great deal of attention because of their intrinsic beauty, and because they converge extremely quickly. For example, *Mathematica* calculates π using a variant of a Ramanujan-type formula due to the Chudnovsky brothers [19]:

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!(13591409+54513013n)}{n!^3(3n)!\,(640320^3)^{n+1/2}}. \tag{3.3}$$

More recent mathematicians including Yang and Zudilin have derived formulas for $1/\pi$ which are not hypergeometric, but still similar to Eq. (3.3). For example, Yang showed that

$$\frac{18}{\pi\sqrt{15}} = \sum_{n=0}^{\infty}\frac{(4n+1)}{36^n}\sum_{k=0}^{n}\binom{n}{k}^4,$$

and Zudilin gave many infinite series for $1/\pi$ containing nested sums of binomial coefficients [20]. All of the formulas that we will prove, including Eq. (3.1), are essentially of this type. Before proving the next theorem, we will point out that equation (3.4) appears implicitly in the work of Chan, Chan and Liu, and can be obtained by combining equations (4.5) and (4.6) in their paper [10].

Theorem 3.1. *For $|u|$ sufficiently small*

$$ {}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1};\frac{108u^2}{(1-4u)^3}\right) = (1-4u)\sum_{n=0}^{\infty}u^n\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2. \tag{3.4}$$

If $|u|$ is sufficiently small

$$ {}_3F_2\left({\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4} \atop 1,1};\frac{256u}{9(1+3u)^4}\right) = \frac{(1+3u)}{(1+u)}\sum_{n=0}^{\infty}\left(\frac{u}{9(1+u)^2}\right)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2. \tag{3.5}$$

Proof. Applying the operator $u\frac{\mathrm{d}}{\mathrm{d}u}$ to Eq. (2.23), and then simplifying yields

$$\sum_{n=0}^{\infty}u^n\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2 = -\frac{(1+32u)}{15(1-16u)}{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1};-\frac{108u}{(1-16u)^3}\right) + \frac{16(1+2u)}{15(1-4u)}{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1};\frac{108u^2}{(1-4u)^3}\right).$$

Eq. (3.4) then follows from applying a ${}_3F_2$ transformation:

$$ {}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1};-\frac{108u}{(1-16u)^3}\right) = \frac{(1-16u)}{(1-4u)}{}_3F_2\left({\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3} \atop 1,1};\frac{108u^2}{(1-4u)^3}\right). \tag{3.6}$$

We can prove Eq. (3.5) in a similar manner by differentiating Eq. (2.24) and then using

$$ {}_3F_2\left({\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4} \atop 1,1};\frac{256u^3}{9(3+u)^4}\right) = \frac{(3+u)}{3(1+3u)}{}_3F_2\left({\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4} \atop 1,1};\frac{256u}{9(1+3u)^4}\right). \tag{3.7}$$

Equation (3.6) can each be derived in three steps. First square both sides of the following ${}_2F_1$ identity (see Corollary 6.2 in [5]):

$$ {}_2F_1\left({\textstyle\frac13,\frac23 \atop 1};\frac{(1-p)(2+p)^2}{4}\right)=\frac{2}{(1+p)}\,{}_2F_1\left({\textstyle\frac13,\frac23 \atop 1};\frac{(1-p)^2(2+p)}{2(1+p)^3}\right). $$

When $|1-p|$ is sufficiently small, we can apply the following ${}_3F_2$ transformation

$$ {}_3F_2\left({\textstyle\frac13,\frac12,\frac23 \atop 1,1};4x(1-x)\right)={}_2F_1{}^2\left({\textstyle\frac13,\frac23 \atop 1};x\right), $$

and then conclude by setting $u=\frac{(-1+p)(3+p)}{16p(2+p)}$. Similarly, equation (3.7) follows from combining another ${}_2F_1$ identity (which can be shown to follow from Theorem 9.15 in [5]):

$$ {}_2F_1\left({\textstyle\frac14,\frac34 \atop 1};1-\frac{64p}{(3+6p-p^2)^2}\right)=\sqrt{\frac{3+6p-p^2}{27-18p-p^2}}\,{}_2F_1\left({\textstyle\frac14,\frac34 \atop 1};1-\frac{64p^3}{(27-18p-p^2)^2}\right), $$

with a similar ${}_3F_2$ transformation

$$ {}_3F_2\left({\textstyle\frac14,\frac12,\frac34 \atop 1,1};4x(1-x)\right)={}_2F_1{}^2\left({\textstyle\frac14,\frac34 \atop 1};x\right), $$

and then setting $u=\frac{9(1-p)}{p(9-p)}$. ■

While the infinite series in Theorem 3.1 are not hypergeometric since they involve nested binomial sums, they are still interesting. In particular, those formulas easily translate into unexpected integrals involving powers of modified Bessel functions. For $|x|$ sufficiently small, Eq. (3.4) is equivalent to

$$ \int_0^\infty e^{-3\left(x+x^{-1}\right)u} I_0^3(2u)\,du=\frac{x}{3\,(1+3x^2)}\,{}_3F_2\left({\textstyle\frac14,\frac12,\frac34 \atop 1,1};\frac{256x^2}{9(1+3x^2)^4}\right), \tag{3.8} $$

where $I_0(u)$ is the modified Bessel function of the first kind. Recall the series expansions for $I_0(2u)$ and $I_0^2(2u)$:

$$ I_0(2u)=\sum_{n=0}^{\infty}\frac{u^{2n}}{n!^2}, \qquad I_0^2(2u)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{u^{2n}}{n!^2}. $$

Eq. (3.8) is surprising because there is no known hypergeometric expression $I_0^3(2u)$ [2]. It is therefore not obvious that the Laplace transform of $I_0^3(2u)$ should equal a hypergeometric function. M. Lawrence Glasser has kindly pointed out that equation (3.8) is essentially a well known result, and that a variety of similar integrals have also been studied by Joyce [14], Glasser and Montaldi [12], and others.

Finally, we will list a few formulas for $1/\pi$. Notice that equation (3.10) first appeared in the work of Chan, Chan and Liu [10].

Corollary 3.2. *Let* $a_n = \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\binom{n}{k}^2$*, then the following formulas are true:*

$$\frac{2}{\pi} = \sum_{n=0}^{\infty}(-1)^n \frac{(3n+1)}{32^n} a_n, \tag{3.9}$$

$$\frac{8\sqrt{3}}{3\pi} = \sum_{n=0}^{\infty} \frac{(5n+1)}{64^n} a_n, \tag{3.10}$$

$$\frac{9+5\sqrt{3}}{\pi} = \sum_{n=0}^{\infty}(6n+3-\sqrt{3})\left(\frac{3\sqrt{3}-5}{4}\right)^n a_n. \tag{3.11}$$

Let $b_n = \binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2$*, then the following identity holds:*

$$\frac{2\left(64+29\sqrt{3}\right)}{\pi} = \sum_{n=0}^{\infty}\left(520n+159-48\sqrt{3}\right)\left(\frac{80\sqrt{3}-139}{484}\right)^n b_n. \tag{3.12}$$

Proof. We can use Eq. (3.4) to easily deduce that if

$$\sum_{n=0}^{\infty}(an+b)\frac{(1/3)_n(1/2)_n(2/3)_n}{n!^3}\left(\frac{108u^2}{(1-4u)^3}\right)^n = \sum_{n=0}^{\infty}(An+B)u^n a_n,$$

then $A = a(1-4u)/(2+4u)$, and $B = a(-4u)(1-4u)/(2+4u) + b(1-4u)$. Since the left-hand side of this last formula equals $1/\pi$ when $\left(a, b, \frac{108u^2}{(1-4u)^3}\right) \in \{\left(\frac{60}{27}, \frac{8}{27}, \frac{2}{27}\right), \left(\frac{2}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{2}\right), \left(\frac{45}{11} - \frac{5}{33}\sqrt{3}, \frac{6}{11} - \frac{13}{99}\sqrt{3}, -\frac{194}{1331} + \frac{225}{2662}\sqrt{3}\right)\}$, it is easy to verify equations (3.9) through (3.11) [11].

We can verify Eq. (3.12) in a similar manner by combining Eq. (3.5) with Ramanujan's formula

$$\frac{8}{\pi} = \sum_{n=0}^{\infty}(20n+3)\frac{(1/4)_n(1/2)_n(3/4)_n}{n!^3}\left(\frac{-1}{4}\right)^n.$$

■

4 Conclusion

We will conclude the paper by suggesting two future projects. Firstly, it would be desirable to determine whether or not a rational series involving b_n exists for $1/\pi$. Secondly, it might be interesting to consider the Mahler measure

$$f_6(u) = \mathrm{m}\left(u - \left(z+z^{-1}\right)^6\left(y+y^{-1}\right)^2(1+x)^3x^{-2}\right),$$

since $f_6(u)$ arises from Ramanujan's theory of signature 6.

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