Functional equations for Mahler measures of genus-one curves

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April 22, 2008

Abstract

In this paper we will establish functional equations for Mahler measures of families of genus-one two-variable polynomials. These families were previously studied by Beauville [3], and their Mahler measures were considered by Boyd [11] and Rodriguez-Villegas [19]. Bertin [8], Zagier [26], and Stienstra [24]. Our functional equations allow us to prove identities between Mahler measures that were conjectured by Boyd. As a corollary, we also establish some new transformations for hypergeometric functions.

1 History and introduction

The goal of this paper is to establish identities between the logarithmic Mahler measures of polynomials with zero varieties corresponding to genus-one curves. Recall that the logarithmic Mahler measure (which we shall henceforth simply refer to as the Mahler measure) of an n-variable Laurent polynomial $P(x_1, x_2, \ldots, x_n)$ is defined by

$$\mathrm{m}\left(P(x_1, \ldots, x_n)\right) = \int_0^1 \cdots \int_0^1 \log \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right) \right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.$$

Many difficult questions surround the special functions defined by Mahler measures of elliptic curves.

The first example of the Mahler measure of a genus-one curve was studied by Boyd [11] and Deninger [13]. Boyd found that

$$\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} L'(E, 0), \tag{1.1}$$

where E denotes the elliptic curve of conductor 15 that is the projective closure of $1 + x + \frac{1}{x} + y + \frac{1}{y} = 0$. As usual, $L(E, s)$ is its L-function, and the question mark above the equals sign indicates numerical equality verified up to 28 decimal places.

Deninger [13] gave an interesting interpretation of this formula. He obtained the Mahler measure by evaluating the Bloch regulator of an element $\{x, y\}$ from a certain K-group. In other words, the Mahler measure is given by a value of an Eisenstein-Kronecker series. Therefore Bloch's and Beilinson's conjectures predict that

$$\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) = cL'(E, 0),$$

where c is some rational number. Let us add that, even if Beilinson's conjectures were known to be true, this would not suffice to prove equality (1.1), since we still would not know the height of the rational number c.

This picture applies to other situations as well. Boyd [11] performed extensive numerical computations within the family of polynomials $k + x + \frac{1}{x} + y + \frac{1}{y}$, as well as within some other genus-one families. Boyd's numerical searches led him to conjecture identities such as

$$\mathrm{m}\left(5 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} 6\mathrm{m}\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right),$$

$$\mathrm{m}\left(8 + x + \frac{1}{x} + y + \frac{1}{y}\right) \stackrel{?}{=} 4\mathrm{m}\left(2 + x + \frac{1}{x} + y + \frac{1}{y}\right).$$

Boyd conjectured conditions predicting when formulas like Eq. (1.1) should exist for the Mahler measures of polynomials with integral coefficients. This was further studied by Rodriguez-Villegas [19] who interpreted these conditions in the context of Bloch's and Beilinson's conjectures. Furthermore, Rodriguez-Villegas used modular forms to express the Mahler measures as Kronecker-Eisenstein series in more general cases. In turn, this allowed him to prove some equalities such as

$$\mathrm{m}\left(4\sqrt{2} + x + \frac{1}{x} + y + \frac{1}{y}\right) = L'\left(E_{4\sqrt{2}}, 0\right), \tag{1.2}$$

$$\mathrm{m}\left(3\sqrt{2} + x + \frac{1}{x} + y + \frac{1}{y}\right) = \frac{5}{2}L'\left(E_{3\sqrt{2}}, 0\right). \tag{1.3}$$

The first equality can be proved using the fact that the corresponding elliptic curve has complex multiplication, and therefore the conjectures are known for this case due to Bloch [10]. The second equality depends on the fact that one has the modular curve $X_0(24)$, and the conjectures then follow from a result of Beilinson.

Rodriguez-Villegas [20] subsequently used the relationship between Mahler measures and regulators to prove a conjecture of Boyd [11]:

$$\mathrm{m}(y^2 + 2xy + y - x^3 - 2x^2 - x) = \frac{5}{7}\mathrm{m}(y^2 + 4xy + y - x^3 + x^2).$$

It is important to point out that he proved this identity without actually expressing the Mahler measures in terms of L-series. Bertin [9] has also proved similar identities using these ideas.

Although the conjecture in Eq. (1.1) remains open, we will in fact prove two of Boyd's other conjectures this paper.

Theorem 1.1. *The following identities are true:*

$$m\left(2+x+\frac{1}{x}+y+\frac{1}{y}\right) = L'\left(E_{3\sqrt{2}},0\right), \tag{1.4}$$

$$m\left(8+x+\frac{1}{x}+y+\frac{1}{y}\right) = 4L'\left(E_{3\sqrt{2}},0\right). \tag{1.5}$$

Our proof of Theorem 1.1 follows from combining two interesting "functional equations" for the function

$$m(k) := \mathrm{m}\left(k+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

Kurokawa and Ochiai [15] recently proved the first functional equation. They showed that if $k \in \mathbb{R}\backslash\{0\}$:

$$m\left(4k^2\right)+m\left(\frac{4}{k^2}\right) = 2m\left(2\left(k+\frac{1}{k}\right)\right). \tag{1.6}$$

In Section 3 we use regulators to give a new proof of Eq. (1.6). We will also prove a second functional equation in Section 2.1 using q-series. In particular, if k is nonzero and $|k|<1$:

$$m\left(2\left(k+\frac{1}{k}\right)\right)+m\left(2\left(\mathrm{i}k+\frac{1}{\mathrm{i}k}\right)\right) = m\left(\frac{4}{k^2}\right). \tag{1.7}$$

Theorem 1.1 follows from setting $k = 1/\sqrt{2}$ in both identities, and then showing that $5m\left(\mathrm{i}\sqrt{2}\right) = 3m\left(3\sqrt{2}\right)$. We have proved this final equality in Section 3.6.

This paper is divided into two sections of roughly equal length. In Section 2 we will prove more identities like Eq. (1.7), which arise from expanding Mahler measures in q-series. In particular, we will look at identities for four special functions defined by the Mahler measures of genus-one curves (see equations (2.1) through (2.4) for notation). Equation (2.14) undoubtedly constitutes the most important result in this part of the paper, since it implies that infinitely many identities like Eq. (1.7) exist. Subsections 2.1 and 2.2 are mostly devoted to transforming special cases of Eq. (2.14) into interesting identities between the Mahler measures of rational polynomials. While the theorems in those subsections rely heavily on Ramanujan's theory of modular equations to alternative bases, we have attempted to maximize readability by eliminating q-series manipulation wherever possible. Finally, we have devoted Subsection 2.3 to proving some useful computational formulas. As a corollary we establish several new transformations for hypergeometric functions, including:

$$\sum_{n=0}^{\infty}\left(\frac{k(1-k)^2}{(1+k)^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}$$
$$=\frac{(1+k)^2}{\sqrt{(1+k^2)\left((1-k-k^2)^2-5k^2\right)}}{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5\left(1+k-k^2\right)}{(1+k^2)^2\left((1-k-k^2)^2-5k^2\right)^2}\right). \tag{1.8}$$

We have devoted Section 3 to further studying the relationship between Mahler measures and regulators. We show how to recover the Mahler measure q-series expansions and the Kronecker-Einsenstein series directly from Bloch's formula for the regulator. This in turn shows that the Mahler measure identities can be viewed as consequences of functional identities for the elliptic dilogarithm.

Many of the identities in this paper can be interpreted from both a regulator perspective, and from a q-series perspective. The advantage of the q-series approach is that it simplifies the process of finding new identities. The fundamental result in Section 2, Eq. (2.14), follows easily from the Mahler measure q-series expansions. Unfortunately the q-series approach does not provide an easy way to explain identities like Eq. (1.6). Unlike most of the other formulas in Section 2, Kurokawa's and Ochiai's result *does not* follow from Eq. (2.14). An advantage of the regulator approach, is that it enables us to construct proofs of both Eq. (1.6) and Eq. (1.7) from a unified perspective. Additionally, the regulator approach seem to provide the only way to prove the final step in Theorem 1.1, namely to show that $5m\left(\mathrm{i}\sqrt{2}\right)=3m\left(3\sqrt{2}\right)$. Thus, a complete view of this subject matter should incorporate both regulator and q-series perspectives.

2 Mahler measures and q-series

In this paper we will consider four important functions defined by Mahler measures:

$$\mu(t) = \mathrm{m}\left(\frac{4}{\sqrt{t}}+x+\frac{1}{x}+y+\frac{1}{y}\right), \tag{2.1}$$
$$n(t) = \mathrm{m}\left(x^3+y^3+1-\frac{3}{t^{1/3}}xy\right), \tag{2.2}$$
$$g(t) = \mathrm{m}\left((x+y)(x+1)(y+1)-\frac{1}{t}xy\right), \tag{2.3}$$
$$r(t) = \mathrm{m}\left((x+y+1)(x+1)(y+1)-\frac{1}{t}xy\right). \tag{2.4}$$

Throughout Section 2 will use the notation $\mu(t)=m\left(\frac{4}{\sqrt{t}}\right)$ for convenience. Recall from [19] and [24], that each of these functions has a simple q-series expansion when t is parameterized correctly. To summarize, if we let $(x;q)_\infty = (1-x)(1-xq)\left(1-xq^2\right)\ldots$, and

$$M(q) = 16q\frac{(q;q)_\infty^8\left(q^4;q^4\right)_\infty^{16}}{(q^2;q^2)_\infty^{24}}, \tag{2.5}$$

$$N(q) = \frac{27q\left(q^3;q^3\right)_\infty^{12}}{(q;q)_\infty^{12}+27q\left(q^3;q^3\right)_\infty^{12}}, \tag{2.6}$$

$$G(q) = q^{1/3}\frac{\left(q;q^2\right)_\infty}{(q^3;q^6)_\infty^3}, \tag{2.7}$$

$$R(q) = q^{1/5}\frac{\left(q;q^5\right)_\infty\left(q^4;q^5\right)_\infty}{\left(q^2;q^5\right)_\infty\left(q^3;q^5\right)_\infty}, \tag{2.8}$$

then for $|q|$ sufficiently small

$$\mu\left(M(q)\right) = -\operatorname{Re}\left[\frac{1}{2}\log(q)+2\sum_{j=1}^{\infty}j\chi_{-4}(j)\log\left(1-q^j\right)\right], \tag{2.9}$$

$$n\left(N(q)\right) = -\operatorname{Re}\left[\frac{1}{3}\log(q)+3\sum_{j=1}^{\infty}j\chi_{-3}(j)\log(1-q^j)\right], \tag{2.10}$$

$$g\left(G^3(q)\right) = -\operatorname{Re}\left[\log(q)+\sum_{j=1}^{\infty}(-1)^{j-1}j\chi_{-3}(j)\log(1-q^j)\right], \tag{2.11}$$

$$r\left(R^5(q)\right) = -\operatorname{Re}\left[\log(q)+\sum_{j=1}^{\infty}j\operatorname{Re}\left[(2-\mathrm{i})\chi_r(j)\right]\log\left(1-q^j\right)\right]. \tag{2.12}$$

In particular, $\chi_{-3}(j)$ and $\chi_{-4}(j)$ are the usual Dirichlet characters, and $\chi_r(j)$ is the character of conductor five with $\chi_r(2) = \mathrm{i}$. We have used the notation $G(q)$ and $R(q)$, as opposed to something like $\tilde{G}(q) = G^3(q)$, in order to preserve Ramanujan's notation. As usual, $G(q)$ corresponds to Ramanujan's cubic continued fraction, and $R(q)$ corresponds to the Rogers-Ramanujan continued fraction [1].

The first important application of the q-series expansions is that they can be used to calculate the Mahler measures numerically. For example, we can calculate $\mu\left(1/10\right)$ with Eq. (2.9), provided that we can first determine a value of q for which $M(q) = 1/10$. Fortunately, the theory of elliptic functions shows that if $\alpha = M(q)$, then

$$q = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}\right). \tag{2.13}$$

Using Eq. (2.13) we easily compute $q = .01975\ldots$, and it follows that $\mu(1/10) = 2.524718\ldots$ The function defined in Eq. (2.13) is called the *elliptic nome*, and is sometimes denoted by $q_2(\alpha)$. Theorem 2.6 provides similarly explicit inversion formulas for Eqs. (2.5) through (2.8).

The second, and perhaps more significant fact that follows from these q-series, is that linear dependencies exist between the Mahler measures. In particular, if

$$f(q) \in \left\{\mu\left(M(q)\right), n\left(N(q)\right), g\left(G^3(q)\right), r\left(R^5(q)\right)\right\},$$

then for an appropriate prime p

$$\sum_{j=0}^{p-1} f\left(\mathrm{e}^{2\pi \mathrm{i} j/p} q\right) = (1+p^2\chi(p)) f\left(q^p\right) - p\chi(p) f\left(q^{p^2}\right), \tag{2.14}$$

where $\chi(j)$ is the character from the relevant q-series. The prime p satisfies the restriction that $p \neq 2$ when $f(q) = g\left(G^3(q)\right)$, and $p \not\equiv 2, 3 \pmod 5$ when $f(q) = r\left(R^5(q)\right)$. The astute reader will immediately recognize that Eq. (2.14) is essentially a Hecke eigenvalue equation. A careful analysis of the exceptional case that occurs when $p = 2$ and $f(q) = g\left(G^3(q)\right)$ leads to the important and surprising inverse relation:

$$\begin{aligned} 3n\left(N(q)\right) &= g\left(G^3\left(q\right)\right) - 8g\left(G^3\left(-q\right)\right) + 4g\left(G^3\left(q^2\right)\right), \\ 3g\left(G^3(q)\right) &= n\left(N(q)\right) + 4n\left(N\left(q^2\right)\right). \end{aligned} \tag{2.15}$$

In the next two subsections we will discuss methods for transforming Eq. (2.14) and Eq. (2.15) into so-called functional equations.

2.1 Functional equations from modular equations

Since the primary goal of this paper is to find relations between the Mahler measures of *rational* (or at least algebraic) polynomials, we will require modular equations to simplify our results. For example, consider Eq. (2.14) when $f(q) = \mu\left(M(q)\right)$ and $p = 2$:

$$\mu\left(M(q)\right) + \mu\left(M(-q)\right) = \mu\left(M\left(q^2\right)\right). \tag{2.16}$$

For our purposes, Eq. (2.16) is only interesting if $M(q)$, $M(-q)$, and $M\left(q^2\right)$ are all simultaneously algebraic. Fortunately, it turns out that $M(q)$ and $M\left(q^2\right)$ (hence also $M(-q)$ and $M\left(q^2\right)$) satisfy a well known polynomial relation.

Definition 2.1. *Suppose that $F(q) \in \{M(q), N(q), G(q), R(q)\}$. An n'th degree modular equation is an algebraic relation between $F(q)$ and $F(q^n)$.*

We will not need to derive any new modular equations in this paper. Berndt proved virtually all of the necessary modular equations while editing Ramanujan's notebooks, see [1], [4], [5], and [6]. Ramanujan seems to have arrived at most of his modular equations through complicated q-series manipulations (of course this is speculation since he did not write down any proofs!). Modular equations involving $M(q)$ correspond to the classical modular equations [5], relations for $N(q)$

correspond to Ramanujan's signature three modular equations [6], and most of the known modular equations for $G(q)$ and $R(q)$ appear in [1].

Now we can finish simplifying Eq. (2.16). Since the classical second-degree modular equation shows that whenever $|q|<1$

$$\frac{4M\left(q^2\right)}{\left(1+M\left(q^2\right)\right)^2}=\left(\frac{M(q)}{M(q)-2}\right)^2,$$

we easily obtain the parameterizations: $M(q)=\frac{4k^2}{(1+k^2)^2}$, $M(-q)=\frac{-4k^2}{(1-k^2)^2}$, and $M\left(q^2\right)=k^4$. Substituting these parametric formulas into Eq. (2.16) yields:

Theorem 2.2. *The following identity holds whenever $|k|<1$:*

$$\begin{aligned}\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right)=&\mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right)\\&+\mathrm{m}\left(2\mathrm{i}\left(k-\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right).\end{aligned}\tag{2.17}$$

We need to make a few remarks about working with modular equations before proving the main theorem in this section. Suppose that for some algebraic function $P(X,Y)$:

$$P\left(F(q),F\left(q^p\right)\right)=0,$$

where $F(q)\in\{M(q),N(q),G(q),R(q)\}$. Using the elementary change of variables, $q\to \mathrm{e}^{2\pi\mathrm{i}j/p}q$, it follows that $P\left(F\left(\mathrm{e}^{2\pi\mathrm{i}j/p}q\right),F\left(q^p\right)\right)=0$ for every $j\in\{0,1,\dots,p-1\}$. If $P(X,Y)$ is symmetric in X and Y, it also follows that $P\left(F\left(q^{p^2}\right),F\left(q^p\right)\right)=0$. Therefore, if $P(X,Y)$ is sufficiently simple (for example a symmetric genus-zero polynomial), we can find simultaneous parameterizations for $F\left(q^p\right)$, $F\left(q^{p^2}\right)$, and $F\left(\mathrm{e}^{2\pi\mathrm{i}j/p}q\right)$ for all j. In such an instance, Eq. (2.14) reduces to an interesting functional equation for one of the four Mahler measures $\{\mu(t),n(t),g(t),r(t)\}$. Five basic functional equations follow from applying these ideas to Eq. (2.14).

Theorem 2.3. *For $|k|<1$ and $k\neq 0$, we have*

$$\mu\left(\frac{4k^2}{\left(1+k^2\right)^2}\right)+\mu\left(\frac{-4k^2}{\left(1-k^2\right)^2}\right)=\mu\left(k^4\right).\tag{2.18}$$

The following identities hold for $|u|$ sufficiently small but non-zero:

$$\begin{aligned}n\left(\frac{27u(1+u)^4}{2\left(1+4u+u^2\right)^3}\right)+&n\left(-\frac{27u(1+u)}{2\left(1-2u-2u^2\right)^3}\right)\\&=2n\left(\frac{27u^4(1+u)}{2\left(2+2u-u^2\right)^3}\right)-3n\left(\frac{27u^2(1+u)^2}{4\left(1+u+u^2\right)^3}\right).\end{aligned}\tag{2.19}$$

If $\zeta_3 = e^{2\pi i/3}$, *and* $Y(t) = 1 - \left(\frac{1-t}{1+2t}\right)^3$, *then*

$$n\left(u^3\right) = \sum_{j=0}^{2} n\left(Y\left(\zeta_3^j u\right)\right). \tag{2.20}$$

If $\zeta_3 = e^{2\pi i/3}$, *and* $Y(t) = t\left(\frac{1-t+t^2}{1+2t+4t^2}\right)$, *then*

$$g(u^3) = \sum_{j=0}^{2} g\left(Y\left(\zeta_3^j u\right)\right). \tag{2.21}$$

If $\zeta_5 = e^{2\pi i/5}$, *and* $Y(t) = t\left(\frac{1-2t+4t^2-3t^3+t^4}{1+3t+4t^2+2t^3+t^4}\right)$, *then*

$$r\left(u^5\right) = \sum_{j=0}^{4} r\left(Y\left(\zeta_5^j u\right)\right). \tag{2.22}$$

Proof. We have already sketched a proof of Eq. (2.18) in the discussion preceding Theorem 2.2.

The proof of Eq. (2.19) requires the second-degree modular equation from Ramanujan's theory of signature three. If $\beta = N\left(q^2\right)$, and $\alpha \in \left\{N(q), N(-q), N\left(q^4\right)\right\}$, then

$$27\alpha\beta(1-\alpha)(1-\beta) - (\alpha+\beta-2\alpha\beta)^3 = 0. \tag{2.23}$$

If we choose u so that $N\left(q^2\right) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$, then we can use Eq. (2.23) to easily verify that $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$, $N(-q) = -\frac{27u(1+u)}{2(1-2u-2u^2)^3}$, and $N\left(q^4\right) = \frac{27u^4(1+u)}{2(2+2u-u^2)^3}$. The proof of Eq. (2.19) follows from applying these parameterizations to Eq. (2.14) when $f(q) = n\left(N(q)\right)$, and $p = 2$.

The proof of Eq. (2.20) requires Ramanujan's third-degree, signature three modular equation. In particular, if $\alpha = N(q)$ and $\beta = N(q^3)$, then

$$\alpha = 1 - \left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3 = Y\left(\beta^{1/3}\right). \tag{2.24}$$

Since $N^{1/3}(q^3) = q \times \{\text{power series in } q^3\}$, a short computation shows that $N(\zeta_3^j q) = Y\left(\zeta_3^j N^{1/3}\left(q^3\right)\right)$ for all $j \in \{0,1,2\}$. Choosing u such that $N(q^3) = u^3$, we must have $N\left(\zeta_3^j q\right) = Y\left(\zeta_3^j u\right)$. Eq. (2.20) follows from applying these parametric formulas to Eq. (2.14) when $f(q) = n\left(N(q)\right)$, and $p = 3$.

Since the proofs of equations (2.21) and (2.22) rely on similar arguments to the proof of Eq. (2.20), we will simply state the prerequisite modular equations. In particular, Eq. (2.21) follows from Ramanujan's third-degree modular equation for the cubic continued fraction. If $\alpha = G(q)$ and $\beta = G\left(q^3\right)$, then

$$\alpha^3 = \beta\left(\frac{1-\beta+\beta^2}{1+2\beta+4\beta^2}\right). \tag{2.25}$$

Similarly, Eq. (2.22) follows from the fifth-degree modular equation for the Rogers-Ramanujan continued fraction. In particular, if $\alpha = R(q)$ and $\beta = R\left(q^5\right)$

$$\alpha^5 = \beta\left(\frac{1-2\beta+4\beta^2-3\beta^3+\beta^4}{1+3\beta+4\beta^2+2\beta^3+\beta^4}\right). \tag{2.26}$$

■

The functional equations in Theorem 2.3 only hold in restricted subsets of $\mathbb{C}$. To explain this phenomenon we will go back to Eq. (2.14). As a general rule, we have to restrict q to values for which *none* of the Mahler measure integrals in Eq. (2.14) vanish on the unit torus. In other words, we can only consider the set of q's for which each term in Eq. (2.14) can be calculated from the appropriate q-series. Next, we may need to further restrict the domain of q depending on where the relevant parametric formulas hold. For example, parameterizations such as $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$ and $N(q^2) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$ hold for $|q|$ sufficiently small, but fail when q is close to 1. After determining the domain of q, we can calculate the domain of u by solving a parametric equation to express u in terms of a q-series.

Theorem 2.4. *For $|p|$ sufficiently small but non-zero*

$$3g(p) = n\left(\frac{27p}{(1+4p)^3}\right) + 4n\left(\frac{27p^2}{(1-2p)^3}\right). \tag{2.27}$$

Furthermore, for $|u|$ sufficiently small but non-zero

$$\begin{aligned} 3n\left(\frac{27u(1+u)^4}{2(1+4u+u^2)^3}\right) =& g\left(\frac{u}{2(1+u)^2}\right) - 8g\left(-\frac{u(1+u)}{2}\right) \\ &+ 4g\left(\frac{u^2}{4(1+u)}\right). \end{aligned} \tag{2.28}$$

Proof. We will prove Eq. (2.28) first. Recall that Eq. (2.15) shows that

$$3n\left(N(q)\right) = g\left(G^3(q)\right) - 8g\left(G^3(-q)\right) + 4g\left(G^3\left(q^2\right)\right).$$

Let us suppose that $q = q_2\left(\frac{u(2+u)^3}{(1+2u)^3}\right)$, where $q_2(\alpha)$ is the elliptic nome. Classical eta function inversion formulas (which we shall omit here) show that for $|u|$ sufficiently small: $G^3(q) = \frac{u}{2(1+u)^2}$, $G^3(-q) = -\frac{u(1+u)}{2}$, $G^3\left(q^2\right) = \frac{u^2}{4(1+u)}$, $N(q) = \frac{27u(1+u)^4}{2(1+4u+u^2)^3}$, and $N\left(q^2\right) = \frac{27u^2(1+u)^2}{4(1+u+u^2)^3}$.

To prove Eq. (2.27) first recall recall that

$$3g\left(G^3(q)\right) = n\left(N(q)\right) + 4n\left(N\left(q^2\right)\right).$$

If we let $p = \frac{u}{2(1+u)^2}$, then it follows that $G^3(q) = p$, $N(q) = \frac{27p}{(1+4p)^3}$, and $N\left(q^2\right) = \frac{27p^2}{(1+2p)^3}$. ■

Theorem 2.4 shows that $g(t)$ and $n(t)$ are essentially interchangeable. In Section 2.3 we will use Eq. (2.27) to derive an extremely useful formula for calculating $g(t)$ numerically.

2.2 Identities arising from higher modular equations

The seven functional equations presented in Section 2.1 are certainly not the only interesting formulas that follow from Eq. (2.14). Rather those results represent the subset of functional equations in which every Mahler measure depends on a rational argument (possibly in a cyclotomic field). If we consider the higher modular equations, then we can establish formulas involving the Mahler measures of the modular polynomials themselves. Eq. (2.32) is the simplest formula in this class of results.

Consider Eq. (2.14) when $p = 3$ and $f(q) = \mu(M(q))$:

$$\sum_{j=0}^{2} \mu\left(M\left(\zeta_3^j q\right)\right) = -8\mu\left(M\left(q^3\right)\right) + 3\mu\left(M\left(q^9\right)\right). \tag{2.29}$$

The third-degree modular equation shows that if $\alpha \in \left\{M(q), M(\zeta_3 q), M\left(\zeta_3^2 q\right), M\left(q^9\right)\right\}$, and $\beta = M\left(q^3\right)$, then

$$G_3(\alpha,\beta) := (\alpha^2+\beta^2+6\alpha\beta)^2 - 16\alpha\beta\left(4(1+\alpha\beta)-3(\alpha+\beta)\right)^2 = 0. \tag{2.30}$$

Since $G_3(\alpha,\beta) = 0$ defines a curve with genus greater than zero, it is impossible to find simultaneous rational parameterizations for all four zeros in α. For example, if we let $\beta = M(q^3) = p(2+p)^3/(1+2p)^3$, then we can obtain the rational expression $M\left(q^9\right) = p^3(2+p)/(1+2p)$, and three messy formulas involving radicals for the other zeros. Despite this difficulty, Eq. (2.29) still reduces to an interesting formula if we recall the factorization

$$G_3\left(\alpha, M\left(q^3\right)\right) = \left(\alpha - M\left(q^9\right)\right)\prod_{j=0}^{2}\left(\alpha - M\left(\zeta_3^j q\right)\right), \tag{2.31}$$

and then use the fact that Mahler measure satisfies $\mathrm{m}(P) + \mathrm{m}(Q) = \mathrm{m}(PQ)$.

Theorem 2.5. *If $G_3(\alpha,\beta)$ is defined in Eq. (2.30), then for $|p|$ sufficiently small but non-zero*

$$\begin{aligned}&\mathrm{m}\left(G_3\left(\frac{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}{16}, \frac{1}{p}\left(\frac{1+2p}{2+p}\right)^3\right)\right)\\ &\quad= -16\log(2) - 16\mu\left(p\left(\frac{2+p}{1+2p}\right)^3\right) + 8\mu\left(p^3\left(\frac{2+p}{1+2p}\right)\right).\end{aligned} \tag{2.32}$$

Proof. First notice that from the elementary properties of Mahler's measure

$$\mu(t) = \frac{1}{2}\mathrm{m}\left(\frac{16}{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2} - t\right) - \frac{1}{2}\log|t|.$$

Applying this identity to Eq. (2.29), and then appealing to Eq. (2.31) yields

$$\mathrm{m}\left(G_3\left(\frac{16}{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}, M\left(q^3\right)\right)\right) = \log\left|M(q)\,M(\zeta_3 q)\,M\left(\zeta_3^2 q\right)M\left(q^9\right)\right| - 16\mu\left(M\left(q^3\right)\right)+8\mu\left(M\left(q^9\right)\right).$$

Elementary q-product manipulations show that $M^4\left(q^3\right) = M(q)\,M(\zeta_3 q)\,M\left(\zeta_3^2 q\right)M\left(q^9\right)$, and since $\alpha^4\beta^4 G_3\left(\frac{1}{\alpha},\frac{1}{\beta}\right) = G_3(\alpha,\beta)$, we obtain

$$\mathrm{m}\left(G_3\left(\frac{\left(x+x^{-1}\right)^2\left(y+y^{-1}\right)^2}{16}, \frac{1}{M\left(q^3\right)}\right)\right) = -16\log(2) - 16\mu\left(M\left(q^3\right)\right) + 8\mu\left(M\left(q^9\right)\right).$$

Finally, if we choose p so that $M\left(q^3\right) = p\left(\frac{2+p}{1+2p}\right)^3$, then $M\left(q^9\right) = p^3\left(\frac{2+p}{1+2p}\right)$, and the theorem follows. ■

Although we completely eliminated the q-series expressions from Eq. (2.32), this is not necessarily desirable (or even possible) in more complicated examples. Consider the identity involving resultants which follows from Eq. (2.14) (and some manipulation) when $p = 11$ and $f(q) = r\left(R^5(q)\right)$:

$$\begin{aligned}\mathrm{m}&\left(\operatorname*{Res}_z\left[z^5 - \frac{xy}{(x+1)(y+1)(x+y+1)}, P\left(z, R^5(q)\right)\right]\right)\\ &= -12\mathrm{m}\left(1+x+y\right) + 12\log\left|R^5(q)\right| + 122r\left(R^5(q)\right) - 11r\left(R^5\left(q^{11}\right)\right).\end{aligned} \tag{2.33}$$

In this formula $P(u,v)$ is the polynomial

$$P(u,v) = uv(1-11v^5-v^{10})(1-11u^5-u^{10}) - (u-v)^{12},$$

which also satisfies $P\left(R(q), R\left(q^{11}\right)\right) = 0$ [21]. Even if rational parameterizations existed for $R(q)$ and $R\left(q^{11}\right)$, substituting such formulas into Eq. (2.33) would probably just make the identity prohibitively complicated.

2.3 Computationally useful formulas, and a few related hypergeometric transformations

While many methods exist for numerically calculating each of the four Mahler measures $\{\mu(t), n(t), g(t), r(t)\}$, two simple and efficient methods are directly related to the material discussed so far.

The first computational method relies on the q-series expansions. For example, we can calculate $\mu(\alpha)$ with Eq. (2.9), provided that a value of q exists for which $M(q) = \alpha$. Amazingly, the elliptic nome function, defined in Eq. (2.13), furnishes a value of q whenever $|\alpha| < 1$. Similar inversion formulas exist for all of the q-products in equations (2.5) through (2.8). Suppose that for $j \in \{2, 3, 4, 6\}$

$$q_j(\alpha) = \exp\left(-\frac{\pi}{\sin\left(\frac{\pi}{j}\right)} \frac{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{j}, 1-\frac{1}{j}; 1; \alpha\right)}\right), \tag{2.34}$$

then we have the following theorem:

Theorem 2.6. *With α and q appropriately restricted, the following table gives inversion formulas for equations* (2.5) *through* (2.8)*:*

α	q
$M(q)$	$q_2(\alpha)$
$N(q)$	$q_3(\alpha)$
$G(q)$	$q_2\left(\frac{u(2+u)^3}{(1+2u)^3}\right)$, *where* $\alpha^3 = \frac{u}{2(1+u)^2}$
$R(q)$	$q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)$, *where* $\alpha^5 = \frac{k(1-k)^2}{(1+k)^2}$

For example: If $|q| < 1$ and $\alpha = M(q)$, then $q = q_2(\alpha)$.

Proof. The inversion formulas for $M(q)$ and $G(q)$ follow from classical eta function identities, and the inversion formula for $N(q)$ follows from eta function identities in Ramanujan's theory of signature three.

The inversion formula for $R(q)$ seems to be new, so we will prove it. Let us suppose that $\alpha = R(q)$ and $k = R(q)R^2\left(q^2\right)$, where q is fixed. A formula of Ramanujan [1] shows that $\alpha^5 = \frac{k(1-k)^2}{(1+k)^2}$, which establishes the second part of the formula. Now suppose that $q = q_2(\alpha_2)$, where $\alpha_2 = M(q)$. A classical identity shows that

$$q\left(-q;q\right)_\infty^{24} = \frac{\alpha_2}{16(1-\alpha_2)^2},$$

and comparing this to Ramanujan's identity

$$q\left(-q;q\right)_\infty^{24} = \left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5,$$

we deduce that

$$\frac{\alpha_2}{(1-\alpha_2)^2} = 16\left(\frac{k}{1-k^2}\right)\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5. \tag{2.35}$$

Now recall that the theory of the signature 4 elliptic nome shows that

$$q = q_2(\alpha_2) = q_4\left(\frac{4\alpha_2}{(1+\alpha_2)^2}\right) = q_4\left(\frac{4\alpha_2/(1-\alpha_2)^2}{1+4\alpha_2/(1-\alpha_2)^2}\right).$$

Substituting Eq. (2.35) into this final result yields

$$q = q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right),$$

which completes the proof.■

The second method for calculating the four Mahler measures, $\{\mu(t), n(t), g(t), r(t)\}$, depends on reformulating them in terms of hypergeometric functions. For example, Rodriguez-Villegas proved [19] the formula

$$\mu(t) = -\frac{1}{2}\mathrm{Re}\left[\log(t/16) + \int_0^t \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;u\right)-1}{u}\mathrm{d}u\right].$$

Translated into the language of generalized hypergeometric functions, this becomes

$$\mu(t) = -\mathrm{Re}\left[\frac{t}{8}{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};t\right) + \frac{1}{2}\log(t/16)\right]. \tag{2.36}$$

He also proved a formula for $n(t)$ which is equivalent to

$$n(t) = -\mathrm{Re}\left[\frac{2t}{27}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};t\right) + \frac{1}{3}\log(t/27)\right]. \tag{2.37}$$

Formulas like Eq. (2.36) and Eq. (2.37) hold some obvious appeal. From a computational perspective they are useful because most mathematics programs have routines for calculating generalized hypergeometric functions. For example, when $|t| < 1$ the Taylor series for the ${}_4F_3$ function easily gives better numerical accuracy than the Mahler measure integrals. Combining Eq. (2.37) with Eq. (2.27) also yields a useful formula for calculating $g(t)$ whenever $|t|$ is sufficiently small:

$$\begin{aligned} g(t) = -\,\mathrm{Re}&\left[\frac{2t}{(1+4t)^3}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27t}{(1+4t)^3}\right) + \frac{8t^2}{(1-2t)^3}{}_4F_3\left({\frac{4}{3},\frac{5}{3},1,1 \atop 2,2,2};\frac{27t^2}{(1-2t)^3}\right)\right.\\ &\left.+\log\left(\frac{t^3}{(1+4t)(1-2t)^4}\right)\right]. \end{aligned} \tag{2.38}$$

So far we have been unable to to find a similar expression for $r(t)$.

Open Problem 2: Express $r(t)$ in terms of generalized hypergeometric functions.

Besides their computational importance, identities like Eq. (2.36) allow for a reformulation of Boyd's conjectures in the language of hypergeometric functions. For example, the conjecture

$$\mathrm{m}\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) \stackrel{?}{=} L'(E,0),$$

where E is an elliptic curve with conductor 15, becomes

$$L'(E,0) \stackrel{?}{=} -2\mathrm{Re}\left[{}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};16\right)\right].$$

A proof of this identity would certainly represent an important addition to the vast literature concerning transformations and evaluations of generalized hypergeometric functions.

In the remainder of this section we will apply our results to deduce a few interesting hypergeometric transformations. For example, differentiating Eq. (2.38) leads to an interesting corollary:

Corollary 2.7. *For $|t|$ sufficiently small*

$$\omega(t) := \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^3 = \frac{1}{1-2t}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27t^2}{(1-2t)^3}\right), \tag{2.39}$$

furthermore

$$\omega\left(\frac{p}{2(1+p)^2}\right) = (1+p)\omega\left(\frac{p^2}{4(1+p)}\right), \tag{2.40}$$

whenever $|p|$ is sufficiently small.

Proof. We can prove Eq. (2.39) by differentiating each side of Eq. (2.38), and then by appealing to Stienstra's formulas [24]. A second possible proof follows from showing that both sides of Eq. (2.39) satisfy the same differential equation.

The shortest proof of Eq. (2.40) follows from a formula due to Zagier [24]:

$$\omega\left(G^3(q)\right) = \prod_{n=0}^{\infty} \frac{\left(1-q^{2n}\right)\left(1-q^{3n}\right)^6}{\left(1-q^n\right)^2\left(1-q^{6n}\right)^3}.$$

First use Zagier's identity to verify that $G^2(q)\omega\left(G^3(q)\right) = G\left(q^2\right)\omega\left(G^3\left(q^2\right)\right)$, and then apply the parameterizations for $G^3(q)$ and $G^3\left(q^2\right)$ from Theorem 2.4.■

We will also make a few remarks about the derivative of $r(t)$. Stienstra has shown that

$$r(t) = -\mathrm{Re}\left[\log(t) + \int_0^t \frac{\phi(u)-1}{u}\mathrm{d}u\right], \tag{2.41}$$

where $\phi(t)$ is defined by

$$\phi(t) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{2.42}$$

Even though we have not discovered a formula for $r(t)$ involving hypergeometric functions, we can still express $\phi(t)$ in terms of the hypergeometric function.

Theorem 2.8. *Let $\phi(t)$ be defined by Eq. (2.42), then for $|k|$ sufficiently small:*

$$\begin{aligned}\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right) =& \frac{(1+k)^2}{\sqrt{(1+k^2)\left((1-k-k^2)^2-5k^2\right)}}\\ &\times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k^5\left(1+k-k^2\right)}{(1+k^2)^2\left((1-k-k^2)^2-5k^2\right)^2}\right),\end{aligned} \tag{2.43}$$

$$\begin{aligned}\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right) =& \frac{(1-k)}{\sqrt{(1+k^2)\left((1+11k-k^2)^2-125k^2\right)}}\\ &\times {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right).\end{aligned} \tag{2.44}$$

Furthermore, $\phi(t)$ satisfies the functional equation:

$$\phi\left(k^2\left(\frac{1+k}{1-k}\right)\right) = \frac{1-k}{(1+k)^2}\phi\left(k\left(\frac{1-k}{1+k}\right)^2\right). \tag{2.45}$$

Proof. We will prove Eq. (2.45) first. A result of Verrill [25] shows that

$$\phi^2\left(R^5(q)\right) = \frac{q}{R^5(q)}\frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty}. \tag{2.46}$$

Combining Eq. (2.46) with the trivial formula $(q^2,q^2)_\infty = (q;q)_\infty(-q;q)_\infty$, we have

$$\frac{\phi^2\left(R^5(q)\right)}{\phi^2\left(R^5(q^2)\right)} = \frac{R^5\left(q^2\right)}{R^5(q)}\frac{\left\{q^{1/24}\left(-q;q\right)_\infty\right\}}{\left\{q^{5/24}\left(-q^5;q^5\right)_\infty\right\}^5}. \tag{2.47}$$

We will apply four of Ramanujan's formulas to finish the proof. If $k=R(q)R^2(q^2)$, then for $|q|$ sufficiently small [1]:

$$R^5(q) = k\left(\frac{1-k}{1+k}\right)^2, \tag{2.48}$$

$$R^5(q^2) = k^2\left(\frac{1+k}{1-k}\right), \tag{2.49}$$

$$q^{1/24}\left(-q;q\right)_\infty = \left(\frac{k}{1-k^2}\right)^{1/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{5/24}, \tag{2.50}$$

$$q^{5/24}\left(-q^5;q^5\right)_\infty = \left(\frac{k}{1-k^2}\right)^{5/24}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^{1/24}. \tag{2.51}$$

Eq. (2.45) follows immediately from substituting these parametric formulas into Eq. (2.47).

Next we will prove Eq. (2.43). Combining Eq. (2.48) with Entry 3.2.15 in [1], we easily obtain

$$q^{5/24}\left(q^5;q^5\right)_\infty=\left\{\frac{k(1-k^2)^2}{(1+k-k^2)\left(1-4k-k^2\right)^2}\right\}^{1/6} q^{1/24}\left(q;q\right)_\infty . \tag{2.52}$$

Now we will evaluate the eta product $q^{1/24}\left(q;q\right)_\infty$. First recall that if $q=q_4(z)$, then

$$q^{1/24}(q;q)_\infty=2^{-1/4}z^{1/24}(1-z)^{1/12}\sqrt{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;z\right)}.$$

In Theorem 2.6 we showed that if $k=R(q)R^2\left(q^2\right)$ then $q=q_4\left(\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)$, hence it follows that

$$\begin{aligned} q^{1/24}\left(q;q\right)_\infty=&\left(\frac{k\left(1-k^2\right)^2\left(1+k-k^2\right)^5\left(1-4k-k^2\right)^{10}}{(1+k^2)^6\left((1+11k-k^2)^2-125k^2\right)^6}\right)^{1/24}\\ &\times\sqrt{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64k\left(1+k-k^2\right)^5}{(1+k^2)^2\left((1+11k-k^2)^2-125k^2\right)^2}\right)}\end{aligned} \tag{2.53}$$

Substituting Eq. (2.53), Eq. (2.52), and Eq. (2.48) into Eq. (2.46) completes the proof of Eq. (2.43). The proof of Eq. (2.44) also follows from an extremely similar argument. ■

We will conclude this section by recording a few formulas which do not appear in [1], but which were probably known to Ramanujan. We will point out that Maier obtained several results along these lines in [17]. Notice that the functional equation for $\phi(t)$ (after substituting $z=k/(1-k^2)$) implies a new hypergeometric transformation:

$$\begin{aligned}\sqrt{\frac{(1+11z)^2-125z^2}{(1-z)^2-5z^2}}\,{}_2F_1&\left(\frac{1}{4},\frac{3}{4};1,\frac{64z^5\left(1+z\right)}{(1+4z^2)\left((1-z)^2-5z^2\right)^2}\right)\\ &={}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{64z\left(1+z\right)^5}{(1+4z^2)\left((1+11z)^2-125z^2\right)^2}\right)\end{aligned} \tag{2.54}$$

Perhaps not surprisingly, we can also use the arguments in this section to deduce that

$$q_4^5\left(\frac{64z\,(1+z)^5}{(1+4z^2)\left((1+11z)^2-125z^2\right)^2}\right)=q_4\left(\frac{64z^5\,(1+z)}{(1+4z^2)\left((1-z)^2-5z^2\right)^2}\right), \tag{2.55}$$

which implies a rational parametrization for the fifth-degree modular equation in Ramanujan's theory of signature 4.

3 A regulator explanation

Now we will reinterpret our identities in terms of the regulators of elliptic curves. The elliptic curves in question are defined by the zero varieties of the polynomials whose Mahler measure we studied. First we will explain the relationship between Mahler measures and regulators. Then we will use regulators to deduce formulas involving Kronecker-Eisenstein series, including equations (2.9), (2.10), (2.11), and (2.12).

We will follow some of the ideas of Rodriguez-Villegas [20].

3.1 The elliptic regulator

Let F be a field. By Matsumoto's Theorem, $K_2(F)$ is generated by the symbols $\{a,b\}$ for $a,b\in F$, which satisfy the bilinearity relations $\{a_1a_2,b\}=\{a_1,b\}\{a_2,b\}$ and $\{a,b_1b_2\}=\{a,b_1\}\{a,b_2\}$, and the Steinberg relation $\{a,1-a\}=1$ for all $a\neq 0$.

Recall that for a field F, with discrete valuation v, and maximal ideal $\mathcal{M}$, the tame symbol is given by

$$(x,y)_v\equiv(-1)^{v(x)v(y)}\frac{x^{v(y)}}{y^{v(x)}} \bmod \mathcal{M}$$

(see [19]). Note that this symbol is trivial if $v(x)=v(y)=0$. In the case when $F=\mathbb{Q}(E)$ (from now on E denotes an elliptic curve), a valuation is determined by the order of the rational functions at each point $S\in E(\bar{\mathbb{Q}})$. We will denote the valuation determined by a point $S\in E(\bar{\mathbb{Q}})$ by v_S.

The tame symbol is then a map $K_2(\mathbb{Q}(E))\to\mathbb{Q}(S)^*$.

We have

$$0\to K_2(E)\otimes\mathbb{Q}\to K_2(\mathbb{Q}(E))\otimes\mathbb{Q}\to\coprod_{S\in E(\bar{\mathbb{Q}})}\mathbb{Q}Q(S)^*\times\mathbb{Q},$$

where the last arrow corresponds to the coproduct of tame symbols.

Hence an element $\{x,y\}\in K_2(\mathbb{Q}(E))\otimes\mathbb{Q}$ can be seen as an element in $K_2(E)\otimes\mathbb{Q}$ whenever $(x,y)_{v_S}=1$ for all $S\in E(\bar{\mathbb{Q}})$. All of the families considered in this paper are tempered according to [19], and therefore they satisfy the triviality of tame symbols.

The regulator map (defined by Beilinson after the work of Bloch) may be defined by

$$\mathtt{r} : K_2(E) \to H^1(E, \mathbb{R})$$
$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$, and

$$\eta(x, y) := \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x.$$

Here we think of $H^1(E, \mathbb{R})$ as the dual of $H_1(E, \mathbb{Z})$. The regulator is well defined because $\eta(x, 1-x) = \mathrm{d}D(x)$, where

$$D(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \arg(1-z) \log|z|$$

is the Bloch-Wigner dilogarithm.

In terms of the general formulation of Beilinson's conjectures this definition is not completely correct. One needs to go a step further and consider $K_2(\mathcal{E})$, where $\mathcal{E}$ is a Néron model of E over $\mathbb{Z}$. In particular, $K_2(\mathcal{E})$ is a subgroup of $K_2(E)$. It seems [19] that a power of $\{x, y\}$ always lies in $K_2(\mathcal{E})$.

Assume that E is defined over $\mathbb{R}$. Because of the way that complex conjugation acts on η, the regulator map is trivial for the classes in $H_1(E, \mathbb{Z})^+$. In particular, these cycles remain invariant under complex conjugation. Therefore it suffices to consider the regulator as a function on $H_1(E, \mathbb{Z})^-$.

We write $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, where τ is in the upper half-plane. Then $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$, where $z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$. Bloch [10] defines the regulator function in terms of a Kronecker-Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi \mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sideset{}{'}\sum_{m,n \in \mathbb{Z}} \frac{\mathrm{e}^{2\pi \mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}, \tag{3.1}$$

where y_τ is the imaginary part of τ.

Let $J(z) = \log|z| \log|1-z|$, and let

$$D(x) = \mathrm{Im}\left(\mathrm{Li}_2(x)\right) + \arg(1-x) \log|x|$$

be the Bloch-Wigner dilogarithm.

Consider the following function on $E(\mathbb{C}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$:

$$J_\tau(z) = \sum_{n=0}^{\infty} J(zq^n) - \sum_{n=1}^{\infty} J(z^{-1}q^n) + \frac{1}{3} \log^2|q| B_3\left(\frac{\log|z|}{\log|q|}\right), \tag{3.2}$$

where $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ is the third Bernoulli polynomial. If we recall that the elliptic dilogarithm is defined by

$$D_\tau(z) := \sum_{n \in \mathbb{Z}} D(zq^n), \tag{3.3}$$

then the regulator function (see [10]) is given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau. \tag{3.4}$$

By linearity, R_τ extends to divisors with support in $E(\mathbb{C})$. Let x and y be non-constant functions on E with divisors

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

Following [10], and the notation in [19], we recall the diamond operation $\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Here $\mathbb{Z}[E(\mathbb{C})]^-$ means that $[-P] \sim -[P]$.

Because R_τ is an odd function, we obtain a map

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{R}.$$

Theorem 3.1. *(Beilinson [3]) $E/\mathbb{R}$ elliptic curve, x, y non-constant functions in $\mathbb{C}(E)$, $\omega \in \Omega^1$*

$$\int_{E(\mathbb{C})} \bar{\omega} \wedge \eta(x,y) = \Omega_0 R_\tau((x) \diamond (y))$$

where Ω_0 is the real period.

Although a more general version of Beilinson's Theorem exists for elliptic curves defined over the complex numbers, the above version has a simpler formulation.

Corollary 3.2. *(after an idea of Deninger) If x and y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, then*

$$-\int_\gamma \eta(x,y) = Im\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where $\Omega = \int_\gamma \omega$.

Proof. Notice that $\mathrm{i}\eta(x,y)$ is an element of the two-dimensional vector space $H^2_{\mathcal{D}}(E(\mathbb{C}), \mathbb{R}(2))$ generated by ω and $\bar{\omega}$. Then we may write

$$\mathrm{i}\eta(x,y) = \alpha[\omega] + \beta[\bar{\omega}],$$

from which we obtain

$$\int_\gamma \mathrm{i}\eta(x,y) = \alpha\Omega + \beta\overline{\Omega}.$$

On the other hand, we have

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x,y) \wedge \bar{\omega} = \alpha \int_{E(\mathbb{C})} \omega \wedge \bar{\omega} = \alpha \mathrm{i} 2 \Omega_0^2 y_\tau,$$

and

$$\int_{E(\mathbb{C})} \mathrm{i}\eta(x,y)\wedge\omega = -\beta \mathrm{i} 2\Omega_0^2 y_\tau.$$

By Beilinson's Theorem

$$\int_\gamma \mathrm{i}\eta(x,y) = -\frac{R_\tau((x)\diamond(y))\Omega}{2\Omega_0 y_\tau} + \frac{\overline{R_\tau((x)\diamond(y))\Omega}}{2\Omega_0 y_\tau},$$

and the statement follows.■

3.2 Regulators and Mahler measure

From now on, we will set $k = \frac{4}{\sqrt{t}}$ in the first family (2.1).

Rodriguez-Villegas [19] proved that if $P_k(x,y) = k + x + \frac{1}{x} + y + \frac{1}{y}$ does not intersect the torus $\mathbb{T}^2$, then

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi}\,\mathtt{r}(\{x,y\})(\gamma). \tag{3.5}$$

Here the $\sim_{\mathbb{Z}}$ stands for "up to an integer number", and γ is a closed path that avoids the poles and zeros of x and y. In particular, γ generates the subgroup $H_1(E,\mathbb{Z})^-$ of $H_1(E,\mathbb{Z})$ where conjugation acts by -1.

We would like to use this property, however we need to exercise caution. In particular, $P_k(x,y)$ intersects the torus whenever $|k| \le 4$ and $k \in \mathbb{R}$. Let us recall the idea behind the proof of Eq. (3.5) for the special case of $P_k(x,y)$. Writing

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

we have

$$m(k) = \mathrm{m}(yP_k(x,y)) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1}(\log^+|y_{(1)}(x)| + \log^+|y_{(2)}(x)|)\frac{\mathrm{d}x}{x}.$$

This last equality follows from applying Jensen's formula with respect to the variable y. When the polynomial does not intersect the torus, we may omit the "+" sign on the logarithm since each $y_{(i)}(x)$ is always inside or outside the unit circle. Indeed, there is always a branch inside the unit circle and a branch outside. It follows that

$$m(k) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1}\log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi}\int_{\mathbb{T}^1}\eta(x,y), \tag{3.6}$$

where $\mathbb{T}^1$ is interpreted as a cycle in the homology of the elliptic curve defined by $P_k(x,y) = 0$, namely $H_1(E,\mathbb{Z})$.

If $k \in [-4,4]$, then we may also assume that $k > 0$ since this particular Mahler measure does not depend on the sign of k. The equation

$$k + x + \frac{1}{x} + y + \frac{1}{y} = 0$$

certainly has solutions when $(x, y) \in \mathbb{T}^2$. However, for $|x| = 1$ and k real, the number $k + x + \frac{1}{x}$ is real, and therefore $y + \frac{1}{y}$ must be real. This forces two possibilities: either y is real or $|y| = 1$. Let $x = \mathrm{e}^{\mathrm{i}\theta}$, then for $-\pi \leq \theta \leq \pi$ we have

$$-k - 2\cos\theta = y + \frac{1}{y}. \tag{3.7}$$

The limiting case occurs when $|k + 2\cos\theta| = 2$. Since we have assumed that k is positive, this condition becomes $k + 2\cos\theta = 2$, which implies that $y = -1$. When $k + 2\cos\theta > 2$ one solution for y, say, $y_{(1)}$, becomes a negative number less than -1, thus $\left|y_{(1)}\right| > 1$ (the other solution $y_{(2)}$ is such that $\left|y_{(2)}\right| < 1$). When $k + 2\cos\theta < 2$, y_i lies inside the unit circle and never reaches 1. What is important is that $\left|y_{(1)}\right| \geq 1$ and $\left|y_{(2)}\right| \leq 1$, so we can still write Eq. (3.6) even if there is a nontrivial intersection with the torus.

3.3 Functional identities for the regulator

First recall a result by Bloch [10] which studies the modularity of R_τ:

Proposition 3.3. *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, *and let* $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. *If we let*

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix},$$

then:

$$R_{\tau'}\left(\mathrm{e}^{2\pi\mathrm{i}(a'+b'\tau')}\right) = \frac{1}{\gamma\bar{\tau}+\delta} R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right).$$

We will need to use some functional equations for J_τ. First recall the following trivial property for $J(z)$:

$$J(z) = p \sum_{x^p = z} J(x). \tag{3.8}$$

Proposition 3.4. *Let p be an odd prime, let $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and let $q_j = \mathrm{e}^{\frac{2\pi\mathrm{i}(\tau+j)}{p}}$ for $j \in \{0, 1, \ldots, p-1\}$. Suppose that $(N, k) = 1$, and $p \equiv \pm 1$ or $0 \pmod N$. Then*

$$(1 + \chi_{-N}(p)p^2) J_{N\tau}(q^k) = \sum_{j=0}^{p-1} p J_{\frac{N(\tau+j)}{p}}(q_j^k) + \chi_{-N}(p) J_{Np\tau}(q^{pk}), \tag{3.9}$$

and for any z we have

$$(\chi_{-N}(p) + p^2) J_{N\tau}(z) = \sum_{j=0}^{p-1} p J_{\frac{N(\tau+j)}{p}}(z) + \chi_{-N}(p) J_{Np\tau}(z). \tag{3.10}$$

Proof. First notice that

$$\sum_{j=0}^{p-1} J_{\frac{N(\tau+j)}{p}}(q_j^k) = \sum_{n=0}^{\infty}\sum_{j=0}^{p-1} J\left(q_j^{Nn+k}\right) - \sum_{n=1}^{\infty}\sum_{j=0}^{p-1} J\left(q_j^{Nn-k}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).$$

By Eq. (3.8) this becomes

$$\begin{aligned}
&= \sum_{\substack{n=0\\ p\nmid Nn+k}}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) - \sum_{\substack{n=1\\ p\nmid Nn-k}}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right) \\
&\quad + \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) - \sum_{n=1}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right) \\
&\quad - \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} \frac{1}{p} J\left(q^{Nn+k}\right) + \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} \frac{1}{p} J\left(q^{Nn-k}\right) \\
&\quad + \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right).
\end{aligned}$$

Rearranging, we find that

$$\begin{aligned}
&= \frac{1}{p} J_{N\tau}\left(q^k\right) - \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right) \\
&\quad - \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} \frac{1}{p} J\left((q^p)^{\frac{Nn+k}{p}}\right) + \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} \frac{1}{p} J\left((q^p)^{\frac{Nn-k}{p}}\right) \\
&\quad + \sum_{\substack{n=0\\ p\mid Nn+k}}^{\infty} pJ\left(q^{\frac{Nn+k}{p}}\right) - \sum_{\substack{n=1\\ p\mid Nn-k}}^{\infty} pJ\left(q^{\frac{Nn-k}{p}}\right) + \frac{4\pi^2 y_\tau^2 N^2}{3p} B_3\left(\frac{k}{N}\right) \\
&= \frac{1}{p} J_{N\tau}\left(q^k\right) - \frac{\chi_{-N}(p)}{p} J_{Np\tau}(q^{pk}) + \chi_{-N}(p) p J_{N\tau}(q^k),
\end{aligned}$$

which proves the assertion.

The second equality follows in a similar fashion. ∎

It is possible to prove analogous identities for D_τ and R_τ.

Proposition 3.5.

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) = J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right) \tag{3.11}$$

Proof. Let $z = \mathrm{e}^{\pi\mathrm{i}\mu}$, then

$$\begin{aligned} J_{2\mu}(z) - J_{2\mu}(-z) =& J(z) - J(-z) \\ &+ \sum_{n=1}^{\infty}\left(J(zq^n) - J(-zq^n) - J(z^{-1}q^n) + J(-z^{-1}q^n)\right) \\ =& \sum_{n=0}^{\infty}\left(J\left(\mathrm{e}^{\pi\mathrm{i}\mu(4n+1)}\right) - J\left(-\mathrm{e}^{\pi\mathrm{i}\mu(4n+1)}\right)\right. \\ &\left. - J\left(\mathrm{e}^{\pi\mathrm{i}\mu(4n+3)}\right) + J\left(-\mathrm{e}^{\pi\mathrm{i}\mu(4n+3)}\right)\right). \end{aligned}$$

On the other hand,

$$J_{\frac{2\mu+1}{2}}(z) = \sum_{n=0}^{\infty}\left(J\left((-1)^n\mathrm{e}^{\pi\mathrm{i}\mu(2n+1)}\right) - J\left((-1)^{n+1}\mathrm{e}^{\pi\mathrm{i}\mu(2n+1)}\right)\right),$$

which proves the equality.■

3.4 The first family

First we will write the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

in Weierstrass form. Consider the rational transformation

$$X = \frac{k+x+y}{x+y} = -\frac{1}{xy}, \qquad Y = \frac{k(y-x)(k+x+y)}{2(x+y)^2} = \frac{(y-x)\left(1+\frac{1}{xy}\right)}{2xy},$$

which leads to

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

It is useful to state the inverse transformation:

$$x = \frac{kX - 2Y}{2X(X-1)}, \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

Notice that E_k contains a torsion point of order 4 over $\mathbb{Q}(k)$, namely $P = \left(1, \frac{k}{2}\right)$. Indeed, this family is the modular elliptic surface associated to $\Gamma_0(4)$.

We can show that $2P = (0,0)$, and $3P = \left(1, -\frac{k}{2}\right)$.

Now

$$(X) = 2(2P) - 2O,$$

and

$$\begin{aligned}(x) =&(2(P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\ =&(P) - (2P) - (3P) + O,\end{aligned}$$

$$\begin{aligned}(y) =&(2(3P) + (2P) - 3O) - (2(2P) - 2O) - ((P) + (3P) - 2O)\\ =&-(P) - (2P) + (3P) + O.\end{aligned}$$

Computing the diamond operation between the divisors of x and y yields

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

Now assume that $k \in \mathbb{R}$ and $k > 4$. We will choose an orientation for the curve and compute the real period. Because P is a point of order 4 and $\int_0^1 \omega$ is real, we may assume that P corresponds to $\frac{3\Omega_0}{4}$.

The next step is to understand the cycle $|x| = 1$ as an element of $H_1(E, \mathbb{Z})$. We would like to compute the value of $\Omega = \int_\gamma \omega$. First recall that

$$\omega = \frac{\mathrm{d}X}{2Y} = \frac{\mathrm{d}x}{x(y - y^{-1})}.$$

In the case when $k > 4$, consider conjugation of ω. This sends $x \to x^{-1}$, and $\frac{\mathrm{d}x}{x} \to -\frac{\mathrm{d}x}{x}$. There is no intersection with the torus, so y remains invariant. Therefore we conclude that Ω is the complex period, and $\frac{\Omega}{\Omega_0} = \tau$, where τ is purely imaginary.

Therefore for k real and $|k| > 4$

$$m(k) = \frac{4}{\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right).$$

Now take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$. By Proposition 3.3

$$R_\tau(-\mathrm{i}) = R_\tau\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4}}\right) = \bar{\tau} R_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right),$$

therefore

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right).$$

If we let $\mu = -\frac{1}{4\tau}$, then for $k \in \mathbb{R}$ we obtain

$$\begin{aligned}m(k) =& -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = \mathrm{Im}\left(\frac{1}{\pi y_\mu} R_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right)\right)\\ =&\mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m + 4\mu n)^2 (m + 4\bar{\mu} n)}\right),\end{aligned}$$

thus recovering a result of Rodriguez-Villegas. We can extend this result to all $k \in \mathbb{C}$, by arguing that both $m(k)$ and $-\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right)$ are the real parts of holomorphic functions that coincide at infinitely many points (see [18]).

Now we will show how to deduce equations (1.7) and (1.6). Applying Eq. (3.9) with $N = 4$, $k = 1$, and $p = 2$, we have

$$J_{4\mu}(q) = 2J_{2\mu}(q_0) + 2J_{2(\mu+1)}(q_1),$$

which translates into

$$\frac{1}{y_{4\mu}} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right) = \frac{1}{y_{2\mu}} J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) + \frac{1}{y_{2\mu}} J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right).$$

This is the content of Eq. (1.7). Setting $\tau = -\frac{1}{2\mu}$, we may also write

$$D_{\frac{\tau}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + D_\tau(-\mathrm{i}\mathrm{e}^{\pi \mathrm{i}\tau}). \tag{3.12}$$

Next we will use Eq. (3.11):

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) = J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right),$$

which translates into

$$\frac{1}{y_{\frac{2\mu+1}{2}}} J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) = \frac{2}{y_{2\mu}} J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - \frac{2}{y_{2\mu}} J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right).$$

Setting $\tau = -\frac{1}{2\mu}$, and using $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ on the left-hand side, we have

$$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - D_\tau\left(-\mathrm{i}\mathrm{e}^{\pi \mathrm{i}\tau}\right). \tag{3.13}$$

Combining equations (3.12) and (3.13), we see that

$$2D_\tau(-\mathrm{i}) = D_{\frac{\tau}{2}}(-\mathrm{i}) + D_{\frac{\tau-1}{2}}(-\mathrm{i}).$$

This is the content of Eq. (1.6).

Similarly, we may deduce Eq. (2.14) from Eq. (3.9) when $k = 1$, $N = 4$, and p is an odd prime.

3.5 A direct approach

It is also possible to prove equations (1.6) and (1.7) directly, without considering the μ-parametrization or the explicit form of the regulator.

For those formulas, it is easy to explicitly write the isogenies at the level of the Weierstrass models. By using the well-known isogeny of degree 2 (see for example [12, 23]):

$$\phi : \{E : y^2 = x(x^2 + ax + b)\} \to \{\widehat{E} : \hat{y}^2 = \hat{x}(\hat{x}^2 - 2a\hat{x} + (a^2 - 4b))\}$$

given by

$$(x, y) \to \left(\frac{y^2}{x^2}, \frac{y(b - x^2)}{x^2}\right)$$

(we require that $a^2-4b\neq 0$), we find

$$\phi_1 : E_{2\left(n+\frac{1}{n}\right)} \to E_{4n^2}, \qquad \phi_2 : E_{2\left(n+\frac{1}{n}\right)} \to E_{\frac{4}{n^2}},$$

$$\phi_1 : (X,Y) \to \left(\frac{X(n^2X+1)}{X+n^2}, -\frac{n^3Y\left(X^2+2n^2X+1\right)}{(X+n^2)^2}\right),$$

$$\phi_2 : (X,Y) \to \left(\frac{X(X+n^2)}{n^2X+1}, -\frac{Y\left(n^2X^2+2X+n^2\right)}{n\left(n^2X+1\right)^2}\right).$$

Let us write x_1, y_1, X_1, Y_1 for the rational functions and $\mathtt{r}_1$ for the regulator in E_{4n^2}, and x_2, y_2, X_2, Y_2, $\mathtt{r}_2$ for the corresponding objects in $E_{\frac{4}{n^2}}$.

It follows that

$$\begin{aligned}\pm m\left(4n^2\right) =& \mathtt{r}_1\left(\{x_1,y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1}\eta(x_1,y_1)\\ =&\frac{1}{4\pi}\int_{|X|=1}\eta(x_1\circ\phi_1, y_1\circ\phi_1)\\ =&\frac{1}{2}\,\mathtt{r}\left(\{x_1\circ\phi_1, y_1\circ\phi_1\}\right),\end{aligned}$$

where the factor of 2 follows from the degree of the isogeny. Similarly, we find that

$$\pm m\left(\frac{4}{n^2}\right) = \mathtt{r}_2\left(\{x_2,y_2\}\right) = \frac{1}{2}\,\mathtt{r}\left(\{x_2\circ\phi_2, y_2\circ\phi_2\}\right).$$

Now we need to compare the values of

$$\mathtt{r}\left(\{x_1\circ\phi_1, y_1\circ\phi_1\}\right), \quad \mathtt{r}\left(\{x_2\circ\phi_2, y_2\circ\phi_2\}\right), \quad \text{and} \quad \mathtt{r}\left(\{x,y\}\right).$$

Recall that $(x)\diamond(y) = 8(P)$, where $P=(1,\frac{k}{2})$. When $k=2\left(n+\frac{1}{n}\right)$, we will also consider the point $Q=\left(-\frac{1}{n^2},0\right)$, which has order 2 (then $P+Q=\left(-1,n-\frac{1}{n}\right)$, $2P+Q=\left(-n^2,0\right)$, etc).

Let P now denote the point in $E_{2\left(n+\frac{1}{n}\right)}$, and let P_1 denote the corresponding point in E_{4n^2}. We have the following table:

$$\phi_1 : \qquad \begin{array}{ccccc} 3P, & P+Q & \to & P_1 \\ 2P, & Q & \to & 2P_1 \\ P, & 3P+Q & \to & 3P_1 \\ O_0, & 2P+Q & \to & O_1 \end{array}.$$

Using this table, and the divisors (x_1) and (y_1) in E_{4n^2}, we can compute $(x_1\circ\phi_1)\diamond(y_1\circ\phi_1)$. We find that

$$(x_1\circ\phi_1)\diamond(y_1\circ\phi_1) = -16(P)+16(P+Q),$$

and similarly

$$(x_2 \circ \phi_2) \diamond (y_2 \circ \phi_2) = -16(P) - 16(P+Q).$$

These computations show that

$$\frac{1}{2}\,\mathtt{r}_0\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right) + \frac{1}{2}\,\mathtt{r}_0\left(\{x_2 \circ \phi_2, y_2 \circ \phi_2\}\right) = 2\,\mathtt{r}_0\left(\{x_0, y_0\}\right), \tag{3.14}$$

and therefore

$$\mathtt{r}_1\left(\{x_1, y_1\}\right) + \mathtt{r}_2\left(\{x_2, y_2\}\right) = 2\,\mathtt{r}_0\left(\{x_0, y_0\}\right). \tag{3.15}$$

We can conclude the proof of Eq. (1.6) by inspecting signs.

To prove Eq. (1.7), it is necessary to use the isomorphism ϕ from Eq. (3.16).

3.6 Relations among $m(2)$, $m(8)$, $m\left(3\sqrt{2}\right)$, and $m\left(\mathrm{i}\sqrt{2}\right)$

Setting $n = \frac{1}{\sqrt{2}}$ in Eq. (1.7), we obtain

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8).$$

Doing the same in Eq. (1.6), we find that

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right).$$

In this section we will establish the identity

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}),$$

from which we can deduce expressions for $m(2)$ and $m(8)$.

Consider the functions f and $1-f$, where $f = \frac{\sqrt{2}Y - X}{2} \in \mathbb{C}(E_{3\sqrt{2}})$. Their divisors are

$$\begin{aligned} \left(\frac{\sqrt{2}Y - X}{2}\right) =& (2P) + 2(P+Q) - 3O, \\ \left(1 - \frac{\sqrt{2}Y - X}{2}\right) =& (P) + (Q) + (3P+Q) - 3O. \end{aligned}$$

The diamond operation yields

$$(f) \diamond (1-f) = 6(P) - 10(P+Q).$$

But $(f) \diamond (1-f)$ is trivial in K-theory, hence

$$6(P) \sim 10(P+Q).$$

Now consider the isomorphism ϕ:

$$\phi : E_{2\left(n+\frac{1}{n}\right)} \to E_{2\left(\mathrm{i}n+\frac{1}{\mathrm{i}n}\right)}, \qquad (X,Y) \to (-X, \mathrm{i}Y) \tag{3.16}$$

This isomorphism implies that

$$\mathtt{r}_{\mathrm{i}\sqrt{2}}(\{x,y\}) = \mathtt{r}_{3\sqrt{2}}(\{x\circ\phi, y\circ\phi\}).$$

But we know that

$$(x\circ\phi)\diamond(y\circ\phi) = 8(P+Q).$$

This implies

$$6\,\mathtt{r}_{3\sqrt{2}}(\{x,y\}) = 10\,\mathtt{r}_{\mathrm{i}\sqrt{2}}(\{x,y\}),$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently, we may conclude that

$$m(8) = \frac{8}{5}m(3\sqrt{2}), \quad m(2) = \frac{2}{5}m(3\sqrt{2}),$$

and finally

$$m(8) = 4m(2).$$

3.7 The Hesse family

We will now sketch the case of the Hesse family:

$$x^3 + y^3 + 1 - \frac{3}{t^{\frac{1}{3}}}xy.$$

This family corresponds to $\Gamma_0(3)$. The diamond operation yields

$$(x)\diamond(y) = 9(P) + 9(A) + 9(B), \tag{3.17}$$

where P is a point of order 3, defined over $\mathbb{Q}(t^{1/3})$, and A, B are points of order 3 such that $A+B+P=O$.

For $0<t<1$, we have

$$n(t) = \frac{9}{2\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau}\left(R_\tau\left(\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\right) + R_\tau\left(\mathrm{e}^{\frac{4\pi\mathrm{i}(1+\tau)}{3}}\right) + R_\tau\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(2+\tau)}{3}}\right)\right)\right).$$

If we let $\mu = -\frac{1}{\tau}$, we obtain, after several steps,

$$n(t) = \mathrm{Re}\left(\frac{27\sqrt{3}y_\mu}{4\pi^2}\sum_{k,n}{}' \frac{\chi_{-3}(n)}{(3\mu k+n)^2(3\bar{\mu}k+n)}\right).$$

Following the previous example, this result may be extended to $\mathbb{C}\setminus\kappa$ by comparing holomorphic functions.

3.8 The $\Gamma_0^0(6)$ example

We will now sketch a treatment of Stienstra's example [24]:

$$(x+1)(y+1)(x+y) - \frac{1}{t}xy.$$

Applying the diamond operation, we have

$$(x) \diamond (y) = -6(P) - 6(2P),$$

where P is a point of order 6.

For t small, one can write

$$g(t) = \frac{3}{\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau}R_\tau(\xi_6^{-1}) + R_\tau(\xi_3^{-1})\right).$$

Eventually, one arrives to

$$g(t) = \mathrm{Re}\left(\frac{36y_\mu}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-3}(m)}{(m+6\mu n)^2(m+6\bar{\mu}n)}\right) + \mathrm{Re}\left(\frac{9y_\mu}{2\pi^2}\sum_{m,n}{}' \frac{\chi_{-3}(m)}{(m+3\mu n)^2(m+3\bar{\mu}n)}\right),$$

thus recovering a result of [24].

3.9 The $\Gamma_0^0(5)$ example

Now we will consider our final example:

$$(x+y+1)(x+1)(y+1) - \frac{1}{t}xy.$$

Applying the diamond operation, we find that

$$(x) \diamond (y) = 10(P) + 5(2P),$$

where P is a torsion point of order 5.

For $t > 0$

$$r(t) = \frac{5}{2\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau}\left(2R_\tau\left(\mathrm{e}^{\frac{8\pi \mathrm{i}}{5}}\right) + R_\tau\left(\mathrm{e}^{\frac{6\pi \mathrm{i}}{5}}\right)\right)\right).$$

Finally,

$$r(t) = -\mathrm{Re}\left(\frac{25\mathrm{i}y_\mu}{4\pi^2}\sum_{m,n}{}' \frac{2\left(\zeta_5^m - \zeta_5^{-m}\right) + \zeta_5^{2m} - \zeta_5^{-2m}}{(m+5\mu n)^2(m+5\bar{\mu}n)}\right).$$

In conclusion, we see that the modular structure comes from the form of the regulator function, and the functional identities are consequences of the functional identities of the elliptic dilogarithm.

4 Conclusion

We have used both regulator and q-series methods to prove a variety of identities between the Mahler measures of genus-one polynomials. We will conclude this paper with a final open problem.

Open Problem 3: How do you characterize all the functional equations of $\mu(t)$?

We have seen that there are identities like Eq. (1.6), stating that

$$2\mathrm{m}\left(2\left(k+\frac{1}{k}\right)+x+\frac{1}{x}+y+\frac{1}{y}\right)=\mathrm{m}\left(4k^2+x+\frac{1}{x}+y+\frac{1}{y}\right)+\mathrm{m}\left(\frac{4}{k^2}+x+\frac{1}{x}+y+\frac{1}{y}\right).$$

While this formula does not follow from Eq. (2.14), it can be proved with regulators.

Indeed, the last section showed us that we can obtain functional identities for the Mahler measures by looking at functional equations for the elliptic dilogarithm.

Now, understanding these identities is a very hard problem. To have an idea of the dimensions of this problem, let us note that equation (3.9) corresponds to the integration of an identity for the Hecke operator T_p. This suggests that more identities will follow from looking at the general operator T_n. And this is just the beginning of the story. . .

Acknowledgements

The authors would like to deeply thank David Boyd and Fernando Rodriguez-Villegas for many helpful discussions, and David Boyd for pointing out the work of Kurokawa and Ochiai [15]. ML extends her gratitude to Christopher Deninger, Herbert Gangl, and Florian Herzig for enlightening discussions.

ML is a postdoctoral fellow at the Pacific Institute for the Mathematical Sciences and the University of British Columbia. This research was also partially conducted while ML was a member at the Institute for Advanced Study, and at the Mathematical Sciences Research Institute, a visitor at the Institut des Hautes Études Scientifiques, a guest at the Max-Planck-Insitut für Mathematik, and she was employed by the Clay Mathematics Institute as a Liftoff Fellow. ML whishes to thank these institutions for their support and hospitality.

This material is partially based upon work supported by the National Science Foundation under agreement No. DMS-0111298.

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