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Partial fractions expansions and identities for products of Bessel functions

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We derive several partial fractions expansions for products of Bessel functions, and use them to prove algebraic relationships between infinite series involving squares of Bessel functions. We also give formulas for the Taylor series coefficients of the zeros of Bessel functions, when the zeros are regarded as functions of the order x of $J_x(y)$. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1866222]

I. INTRODUCTION

In this paper we derive several partial fractions expansions for products of Bessel functions, the simplest of which is given by

$$J_x(y)J_{-x}(y)\frac{\pi}{\sin(\pi x)} = \sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+x}.$$

These formulas, which are presented in Sec. II, allow us to easily characterize $J_x(y)J_{-x}(y)$ as a function of x. Surprisingly, these sorts of identities are generally overlooked in the modern literature on Bessel functions.

We can apply these formulas to study the zeros of Bessel functions. Let $\nu_k(x)$ denote the kth zero of $J_x(y)$. In Sec. IV we recursively compute the Taylor series coefficients of $\nu_k(x)$ at the half-integers. In Sec. V we discuss computing the Taylor series at zero and the positive integers. The form that we give for the coefficients is interesting because they are expressed as polynomials in functions defined by series of squares of Bessel functions.

Besides appearing in the Taylor series coefficients of $\nu_k(x)$, functions like

$$\mu_k(y) = \sum_{n=1}^{\infty}\frac{J_n^2(y)}{n^k}, \quad \beta_k(y) = \sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{(2n+1)^k},$$

obey a bewildering variety of identities and algebraic relations. In Secs. IV, V, and VII we prove many interesting formulas involving these functions. We also present several associated open problems at the end of Secs. IV–VI.

Throughout this paper we will assume the usual differentiation and addition formulas for Bessel functions,

$$2\frac{x}{y}J_x(y) = J_{x-1}(y) + J_{x+1}(y),$$

$$2\frac{d}{dy}J_x(y) = J_{x-1}(y) - J_{x+1}(y).$$

We will also employ a standard Wronskian relation,

$$J_{x-1}(y)J_{-x}(y) + J_{1-x}(y)J_x(y) = \frac{2\sin(\pi x)}{\pi y}.$$

II. IDENTITIES AND PARTIAL FRACTIONS EXPANSIONS FOR BESSEL FUNCTIONS

The Bessel function of the first kind of order x is defined by the infinite series

$$J_x(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+x+1)} \left(\frac{y}{2}\right)^{2n+x}, \tag{2.1}$$

where $\Gamma(x)$ is the usual gamma function.

We will use the following three partial fractions expansions throughout this paper.

Proposition 2.1: The following formulas hold for all $y \in \mathbf{C}$ and for all $x \notin \mathbf{Z}$,

$$J_x(y) = \frac{\left(\frac{y}{2}\right)^x}{\Gamma(x)} \sum_{n=0}^{\infty} \frac{J_n(y)}{n!} \frac{\left(\frac{y}{2}\right)^n}{n+x}, \tag{2.2}$$

$$J_x(y)J_{-x}(y)\frac{\pi}{\sin(\pi x)} = \sum_{n=-\infty}^{\infty} \frac{J_n^2(y)}{n+x}, \tag{2.3}$$

$$J_x(y)J_{1-x}(y)\frac{\pi}{\sin(\pi x)} = \sum_{n=-\infty}^{\infty} \frac{J_n(y)J_{n+1}(y)}{n+x}. \tag{2.4}$$

Equation (2.2) has a long history. It appears in several 19th century and 20th century books, including Lommel,[2] Schafheitlin,[4] and Watson (Ref. 5, p.143). From (2.2) one can easily derive Lommel's expression for $Y_0(z)$,

$$Y_0(z) = \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{J_n(z)}{n!} \frac{\left(\frac{z}{2}\right)^n}{n}. \tag{2.5}$$

The function $Y_0(z)$ is a Bessel function of the second kind (Ref. 5, p.64).

Equations (2.3) and (2.4) seem to be omitted from the modern literature, as well as all of the easily accessible 19th century books. Considering their relative simplicity, it seems unlikely that 19th century mathematicians would have missed them. In fact, the following integral (Ref. 1, p.756)

$$\int_0^{\pi/2} \cos(2xu)J_0(2y\cos(u))\mathrm{d}u = \frac{\pi}{2}J_x(y)J_{-x}(y) \tag{2.6}$$

is exactly equivalent to Eq. (2.3).

We have found a particularly simple way to prove formulas (2.3) and (2.4) that is worth relating.

Proof: Lommel proves Eq. (2.2) in Ref. 2. We can also prove this result by applying the following elementary partial fractions expansion to Eq. (2.1):

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \sum_{j=0}^{n} \frac{(-1)^j}{j+x}\binom{n}{j}.$$

To prove Eq. (2.4) we will use the reflection formula for the gamma function,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

and we will apply formula (2.2) several times. Proceeding with the proof,

$$J_x(y)J_{1-x}(y)\frac{\pi}{\sin(\pi x)} = (J_x(y)\Gamma(x))(J_{1-x}(y)\Gamma(1-x)) = \left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^{n+x}}{n+x}\right)\left(\sum_{m=0}^{\infty}\frac{J_m(y)}{m!}\frac{\left(\frac{y}{2}\right)^{m+1-x}}{m+1-x}\right)$$

$$= \sum_{n\geqslant 0, m\geqslant 0}^{\infty}\frac{J_n(y)J_m(y)}{n!m!}\frac{\left(\frac{y}{2}\right)^{n+m+1}}{(n+x)(m+1-x)}$$

$$= \sum_{n\geqslant 0, m\geqslant 0}^{\infty}\frac{J_n(y)J_m(y)}{n!m!}\frac{\left(\frac{y}{2}\right)^{n+m+1}}{n+m+1}\left(\frac{1}{n+x}+\frac{1}{m+1-x}\right).$$

Next split the sum into two pieces, then rearrange the order of summation to get

$$= \sum_{n=0}^{\infty}\frac{J_n(y)}{n!(n+x)}\left(\sum_{m=0}^{\infty}\frac{J_m(y)}{m!}\frac{\left(\frac{y}{2}\right)^{n+m+1}}{n+m+1}\right) + \sum_{m=0}^{\infty}\frac{J_m(y)}{m!(m+1-x)}\left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^{n+m+1}}{n+m+1}\right).$$

Observe that the two nested sums in the preceding equation are just special cases of Eq. (2.2). Substituting the appropriate expressions we have

$$= \sum_{n=0}^{\infty}\frac{J_n(y)}{n!(n+x)}\Gamma(n+1)J_{n+1}(y) + \sum_{m=0}^{\infty}\frac{J_m(y)}{m!(m+1-x)}\Gamma(m+1)J_{m+1}(y)$$

$$= \sum_{n=0}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x} + \sum_{m=0}^{\infty}\frac{J_m(y)J_{m+1}(y)}{m+1-x}.$$

Now let $m+1=-n$, and change the indices of summation on the right-hand sum to get

$$= \sum_{n=0}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x} + \sum_{n=-1}^{-\infty}\frac{J_{-n-1}(y)J_{-n}(y)}{-n-x}.$$

Finally, recall that if $m\in\mathbf{Z}$ then $J_{-m}(y)=(-1)^m J_m(y)$. Substituting this relation into the preceding equation yields

$$= \sum_{n=0}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x} + \sum_{n=-1}^{-\infty}\frac{J_n(y)J_{n+1}(y)}{n+x} = \sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x},$$

completing the proof of (2.4).

The proof of Eq. (2.3) is nearly identical to the above proof, except that slightly more care must be taken when combining the partial fractions expansions for $J_x(y)$ and $J_{-x}(y)$. ■

Proposition 2.2: The infinite sums appearing in Proposition 2.1 are related as follows:

$$\left(\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+x}\right)\left(\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+1-x}\right) = \frac{2}{y}\left(\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x}\right) - \left(\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x}\right)^2, \tag{2.7}$$

$$\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x}=\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+1-x}, \tag{2.8}$$

$$\left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^n}{n+x}\right)\left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^{n+1}}{n+1-x}\right)=\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x}, \tag{2.9}$$

$$\left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^n}{n+x}\right)\left(\sum_{n=0}^{\infty}\frac{J_n(y)}{n!}\frac{\left(\frac{y}{2}\right)^{n}}{n-x}\right)=\frac{-1}{x}\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+x}. \tag{2.10}$$

Proof: Equations (2.8)–(2.10) follow trivially from Proposition 2.1.

Equation (2.7) is a simple consequence of the Wronskian relation for Bessel functions,

$$J_{x-1}(y)J_{-x}(y)+J_{1-x}(y)J_x(y)=\frac{2\sin(\pi x)}{\pi y}.$$

We will prove (2.7) as follows. Observe from Proposition 2.1 that the left-hand side (LHS) of Eq. (2.7) is given by

$$\text{LHS}=J_x(y)J_{1-x}(y)J_{x-1}(y)J_{-x}(y)\left(\frac{\pi}{\sin(\pi x)}\right)^2,$$

and the right-hand side (RHS) is given by

$$\text{RHS}=J_x(y)J_{1-x}(y)\left(\frac{\pi}{\sin(\pi x)}\right)^2\left(\frac{2\sin(\pi x)}{\pi y}-J_x(y)J_{1-x}(y)\right).$$

Apply the Wronskian relation to the RHS to see that LHS=RHS, which establishes (2.7). ■

Of course these are not the only partial fractions expansions available for Bessel functions. There are multitudes of ways to generalize these results. We state several generalizations in the next proposition.

Proposition 2.3: If x, y, and z are complex numbers such that $x \notin \mathbf{Z}$, then

$$\frac{\pi}{\sin(\pi x)}J_{z-x}(y)J_{x-z}(y)=\sum_{n=-\infty}^{\infty}(-1)^n\frac{J_{z+n}(y)J_{-z-n}(y)}{x+n}, \tag{2.11}$$

$$\frac{\pi}{\sin(\pi x)}J_x(y)J_{-x}(z)=\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_n(z)}{n+x}\left(\frac{y}{z}\right)^{n+x}, \tag{2.12}$$

$$\frac{\pi}{\sin(\pi x)}J_x(y)J_{1-x}(z)=\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(z)}{n+x}\left(\frac{y}{z}\right)^{n+x}. \tag{2.13}$$

Suppose that $x_1+x_2+x_3=1$, then

$$J_{x_1}(y)J_{x_2}(y)J_{x_3}(y)\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)=\sum_{n\geqslant 0,m\geqslant 0}^{\infty}J_n(y)J_m(y)J_{n+m+1}(y)\binom{n+m}{m}\left(\frac{1}{(n+x_1)(m+x_2)}+\frac{1}{(n+x_1)(m+x_3)}+\frac{1}{(n+x_2)(m+x_3)}\right). \tag{2.14}$$

Letting $x_1=x_2=x_3=\frac{1}{3}$ in Eq. (2.14) yields an interesting double series for $(J_{1/3}(y))^3$,

$$(J_{1/3}(y))^3 = \frac{1}{\Gamma^3(4/3)} \sum_{n\geqslant 0, m\geqslant 0}^{\infty} \frac{J_n(y)J_m(y)J_{n+m+1}(y)}{(3n+1)(3m+1)} \binom{n+m}{m}. \tag{2.15}$$

III. PROPERTIES OF THE FUNCTION $\nu_k(x)$

We will give a brief description of the function $\nu_k(x)$, and then prove a new form for the differential equation that $\nu_k(x)$ satisfies. Throughout this section we will assume that $x\in\mathbf{R}$.

It is well known that $J_x(y)$ has infinitely many real zeros, and a finite number of complex zeros. We can regard the zeros of $J_x(y)$ as functions of the order x.

Definition 3.1: Let $\nu_k(x)$ denote the kth *real zero of $J_x(y)$ to the right of x. Then $J_x(\nu_k(x))=0$ for all x.*

It can be shown that $\nu_k(x)$ is continuous and differentiable for all $x\in\mathbf{R}\setminus\{-1,-2,\ldots\}$. In fact, it is well known that $\nu_k(x)$ satisfies transcendental differential equations. Nicholson's differential equation (Ref. 5, p. 508) is given by

$$\nu_k'(x) = -2\nu_k(x)\int_0^\infty K_0(2\nu_k(x)\sinh(t))e^{-2xt}\,\mathrm{d}t, \tag{3.1}$$

and is valid provided that $x\notin\{-1,-2,\ldots\}$. The negative integers must be excluded from the domain of $\nu_k(x)$, because $\nu_k(x)$ is discontinuous at each of these points. These discontinuities occur because new solution curves of $J_x(y)=0$ come into existence when $x\in\{-1,-2,\ldots\}$. Watson provides a nice illustration (Ref. 5, p. 510) of this phenomenon.

The form of the differential equation that we are interested in follows almost directly from formula (2.3). Proposition 3.2 will allow us to express derivatives of $\nu_k(x)$ in terms of infinite series that obey interesting algebraic relations [see Theorem 4.7 and Eq. (4.24)].

Proposition 3.2: If $x\notin\{-1,-2,\ldots\}$, then $\nu_k(x)$ satisfies the differential equation

$$\nu_k'(x) = \frac{\nu_k(x)}{2}\sum_{n=-\infty}^{\infty}\frac{J_n^2(\nu_k(x))}{(n+x)^2}. \tag{3.2}$$

Proof: This proof is quite standard (Ref. 5, p. 507). By the definition of $\nu_k(x)$, and by formula (2.3) we have

$$0 = \sum_{n=-\infty}^{\infty}\frac{J_n^2(\nu_k(x))}{n+x}.$$

Now assume that x is not a negative integer, and take the derivative of each side to show that

$$0 = -\sum_{n=-\infty}^{\infty}\frac{J_n^2(\nu_k(x))}{(n+x)^2} + \nu_k'(x)\frac{\mathrm{d}}{\mathrm{d}\nu}\sum_{n=-\infty}^{\infty}\frac{J_n^2(\nu_k(x))}{(n+x)}.$$

Substituting Eq. (2.3) for the sum on the right-hand side we find that

$$0 = -\sum_{n=-\infty}^{\infty}\frac{J_n^2(\nu_k(x))}{(n+x)^2} + \nu_k'(x)\frac{\pi}{\sin(\pi x)}\frac{\mathrm{d}}{\mathrm{d}\nu}(J_x(\nu_k(x))J_{-x}(\nu_k(x))). \tag{3.3}$$

Now expand and simplify $\mathrm{d}/\mathrm{d}\nu(J_x(\nu_k(x))J_{-x}(\nu_k(x)))$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\nu}(J_x(\nu_k(x))J_{-x}(\nu_k(x))) = \frac{J_{x-1}(\nu_k(x)) - J_{x+1}(\nu_k(x))}{2}J_{-x}(\nu_k(x)) + J_x(\nu_k(x))\frac{\mathrm{d}}{\mathrm{d}\nu}(J_{-x}(\nu_k(x)))$$
$$= \frac{J_{x-1}(\nu_k(x)) - J_{x+1}(\nu_k(x))}{2}J_{-x}(\nu_k(x)) + 0.$$

If $\nu_k(x) \neq 0$, then the Bessel function addition formula shows that $J_{x+1}(\nu_k(x)) = -J_{x-1}(\nu_k(x))$. Since the only x values for which we may have $\nu_k(x)=0$ are the negative integers (Ref. 5, p. 510), the initial assumption that $x \notin \{-1,-2,\ldots\}$ satisfies this condition. Therefore we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\nu}(J_x(\nu_k(x))J_{-x}(\nu_k(x))) = J_{x-1}(\nu_k(x))J_{-x}(\nu_k(x)).$$

The Wronskian relation shows that

$$J_{x-1}(\nu_k(x))J_{-x}(\nu_k(x)) = \frac{2\sin(\pi x)}{\pi\nu_k(x)},$$

which allows us to evaluate $(\mathrm{d}/\mathrm{d}\nu)(J_x(\nu_k(x))J_{-x}(\nu_k(x)))$ in a simple form,

$$\frac{\mathrm{d}}{\mathrm{d}\nu}(J_x(\nu_k(x))J_{-x}(\nu_k(x))) = \frac{2\sin(\pi x)}{\pi\nu_k(x)}.$$

Substituting this final result into Eq. (3.3) completes the proof. ■

For more technical results about the zeros of Bessel functions, Muldoon's paper[3] provides an excellent list of references.

IV. THE TAYLOR SERIES FOR $\nu_k(x)$ AT THE HALF-INTEGERS

In this section we will give a recursive formula for computing the derivatives of $\nu_k(x)$ at the half-integers. The resulting Taylor series converge relatively slowly, however the form that we give for the coefficients is very interesting. We show that the Taylor coefficients can always be expressed in terms of polynomials of entire functions. The entire functions in question are defined by summations of reciprocal powers of integers, with squares of Bessel functions in the numerators.

Definition 4.1: Let k be an integer, and define $\beta_k(y)$ *and* $d_k(y)$ *by*

$$\beta_k(y) = \sum_{n=-\infty}^{\infty} \frac{J_n^2(y)}{(2n+1)^k},$$

$$d_k(y) = \sum_{n=-\infty}^{\infty} \frac{J_n(y)J_{n+1}(y)}{(2n+1)^k}.$$

Also define the more general function

$$\beta_k^{(a,b)}(y) = \sum_{n=-\infty}^{\infty} \frac{J_{n+a}(y)J_{n+b}(y)}{(2n+1)^k}.$$

Proposition 4.2: The functions $\beta_k(y)$ *and* $d_k(y)$ *have the following elementary properties:*

$$\beta_0(y) = 1, \tag{4.1}$$

$$\beta_1(y) = \frac{\sin(2y)}{2y}, \tag{4.2}$$

$$\beta_2(y) = \frac{1}{2y}\int_0^{2y} \frac{\sin(z)}{z}\mathrm{d}z, \tag{4.3}$$

$$d_{2k}(y) = 0 \quad \text{if } k \geqslant 0 \text{ is an integer}, \tag{4.4}$$

$$d_1(y) = \frac{1-\cos(2y)}{2y}. \tag{4.5}$$

Since (4.5) will be used later, we note that it follows from (2.4).

As a simple corollary to Proposition 4.2 we can obtain formulas for $1/\pi$. Here are two examples,

$$\frac{1}{\pi} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{J_n^2\left(\frac{\pi}{4}\right)}{2n+1}, \tag{4.6}$$

$$\frac{1}{\pi} = \frac{4}{9}\sum_{n=-\infty}^{\infty} \frac{J_n\left(\frac{\pi}{3}\right)J_{n+1}\left(\frac{\pi}{3}\right)}{2n+1}. \tag{4.7}$$

Proposition 4.3: Let $k \geqslant 1$ *be an integer, then* $\beta_k^{(a,b)}(y)$ *has the following elementary properties for every pair* $(a,b) \in \mathbf{Z}^2$:

$$\beta_k^{(a,b)}(y) = \frac{1}{y}\beta_{k-1}^{(a-1,b)}(y) + \frac{2a-3}{y}\beta_k^{(a-1,b)}(y) - \beta_k^{(a-2,b)}(y), \tag{4.8}$$

$$\beta_k^{(a,b)}(y) = \beta_k^{(b,a)}(y), \tag{4.9}$$

$$\beta_0^{(a,b)}(y) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases} \tag{4.10}$$

$$\beta_k^{(0,0)}(y) = \beta_k(y), \tag{4.11}$$

$$\beta_k^{(0,1)}(y) = d_k(y), \tag{4.12}$$

$$\beta_k^{(1,1)}(y) = (-1)^k\beta_k(y). \tag{4.13}$$

Proposition 4.3 establishes linear dependencies between $\beta_k^{(a,b)}(y)$, $\beta_k(y)$, and $d_k(y)$ for all $(a,b) \in \mathbf{Z}^2$. To express $\beta_k^{(a,b)}(y)$ in terms of $d_m(y)$'s and $\beta_m(y)$'s, use the recursion formula (4.8) while applying formula (4.10) as many times as necessary. For example, we can show that

$$\beta_k^{(-1,-1)}(y) = \left(\frac{1}{y^2} + (-1)^k\right)\beta_k(y) - \frac{2}{y^2}\beta_{k-1}(y) + \frac{1}{y^2}\beta_{k-2}(y) - \frac{2}{y}d_{k-1}(y) + \frac{2}{y}d_k(y). \tag{4.14}$$

The next proposition shows that $d_k(y)$ can always be written as a polynomial in $\beta_m(y)$'s for $m \leqslant k$, and the elementary functions. As a result we can always express $\beta_k^{(a,b)}(y)$ as a polynomial in $\beta_2(y), \beta_3(y), \ldots, \beta_k(y)$, and elementary functions.

Proposition 4.4: We can calculate $d_{2n+1}(y)$ *recursively using the following formula:*

$$\frac{\cos(2y)}{y}d_{2n+1}(y) = \sum_{k=1}^{2n-1} d_{k+1}(y)d_{2n+1-k}(y) + \sum_{k=0}^{2n}(-1)^k\beta_{k+1}(y)\beta_{2n+1-k}(y). \tag{4.15}$$

Proof: We will use Eq. (2.7) to prove this result. First observe that a slight rearrangement of (2.7) yields

$$\left(\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+x}\right)\left(\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+1-x}\right) = \frac{1}{y^2} - \left(\frac{1}{y} - \sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x}\right)^2. \tag{4.16}$$

If $|x-\frac{1}{2}|<\frac{1}{2}$, then we can expand each infinite series about $x=1/2$ to obtain

$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+x} = \sum_{k=0}^{\infty}(-1)^k 2^{k+1}\beta_{k+1}(y)(x-1/2)^k,$$

$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(y)}{n+1-x} = \sum_{k=0}^{\infty}2^{k+1}\beta_{k+1}(y)(x-1/2)^k,$$

$$\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{n+x} = \sum_{k=0}^{\infty}2^{2k+1}d_{2k+1}(y)(x-1/2)^{2k}.$$

Substituting these formulas into (4.16), then collecting the series coefficients on each side of the equality proves Eq. (4.15). ■

Applying Proposition 4.4 yields the following evaluations of $d_3(y)$ and $d_5(y)$:

$$d_3(y) = \tan(2y)\beta_3(y) - \frac{y}{\cos(2y)}\beta_2^2(y), \tag{4.17}$$

$$d_5(y) = \tan(2y)\beta_5(y) - \frac{2y}{\cos(2y)}\beta_4(y)\beta_2(y) + \frac{y}{\cos^3(2y)}\beta_3^2(y) - 2y^2\frac{\sin(2y)}{\cos^3(2y)}\beta_3(y)\beta_2^2(y) + \frac{y^3}{\cos^3(2y)}\beta_2^4(y). \tag{4.18}$$

Now that we have shown how to write $d_k(y)$ and $\beta_k^{(a,b)}(y)$ in terms of $\beta_k(y)$, it is necessary to justify the claim that all of these functions are entire. Considering the fact that formula (4.17) contains multiple terms involving $1/\cos(2y)$, this claim is not obvious.

Theorem 4.5: *The functions $\beta_k(y)$, $d_k(y)$, and $\beta_k^{(a,b)}(y)$ are entire functions for all $k\geqslant 0$, and for all $(a,b)\in\mathbf{Z}^2$.*

Proof: Let $E_k(y)$ denote the kth Euler polynomial. Recall that $E_k(y)$ is a kth degree polynomial with generating function

$$\frac{2e^{yt}}{e^t+1} = \sum_{k=0}^{\infty}E_k(y)\frac{t^k}{k!}.$$

Then for all $k\geqslant 0$, and for all $(a,b)\in\mathbf{Z}^2$, we have the following representation of $\beta_{k+1}^{(a,b)}(y)$:

$$\beta_{k+1}^{(a,b)}(y) = (-1)^a\frac{\pi^k}{k!}\int_0^{\pi/2}\cos\left((a+b-1)z+\frac{\pi k}{2}\right)J_{b-a}(2y\cos z)E_k\left(\frac{1}{2}+\frac{z}{\pi}\right)dz. \tag{4.19}$$

Equation (4.19) expresses $\beta_k^{(a,b)}(y)$ as a finite integral of entire functions, so we conclude that $\beta_k^{(a,b)}(y)$ is entire for all $k \geqslant 1$ and for all $(a,b) \in \mathbf{Z}^2$. The $k=0$ case is clear from Eq. (4.10). This also proves that $\beta_k(y)$ and $d_k(y)$ are entire, since they are just special cases of $\beta_k^{(a,b)}(y)$. Recall that $\beta_k(y)=\beta_k^{(0,0)}(y)$ and $d_k(y)=\beta_k^{(0,1)}(y)$.

The proof of Eq. (4.19) is an exercise that we will leave to the reader. ■

The next proposition shows that $(d/dy)\beta_k(y)$ and $(d/dy)d_k(y)$ are just linear combinations of $d_k(y)$, $\beta_k(y)$, and $\beta_{k-1}(y)$ (with coefficients involving y and $1/y$). While Proposition 4.4 allows us to eliminate the notation $d_k(y)$, the convenience of working with linear differential equations justifies its continued use.

Proposition 4.6: The derivatives of $\beta_k(y)$ and $d_k(y)$ are related as follows:

$$\frac{\mathrm{d}}{\mathrm{d}y}(y\beta_k(y)) = \beta_{k-1}(y) - (1-(-1)^k)y d_k(y),$$

$$\frac{\mathrm{d}}{\mathrm{d}y}(y d_k(y)) = (1-(-1)^k)y\beta_k(y). \tag{4.20}$$

Proposition 4.6 allows us to reformulate our study of $\beta_k(y)$ and $d_k(y)$ from a differential equations perspective. By substituting the appropriate formulas for $d_k(y)$ into Eq. (4.20), we can express derivatives of $y\beta_k(y)$ as polynomials in $\beta_2(y), \beta_3(y), \ldots, \beta_k(y)$ and the elementary functions. For example, if we plug (4.17) into (4.20) we find

$$\frac{\mathrm{d}}{\mathrm{d}y}(y\beta_3(y)) = \beta_2(y) - 2\tan(2y)y\beta_3(y) + 2\sec(2y)(y\beta_2(y))^2. \tag{4.21}$$

Solving this resulting system of nonlinear differential equations will naturally lead to some very complicated integrals. To solve Eq. (4.21) for $\beta_3(y)$, multiply each side by $\sec(2y)$ and collect the terms involving $\beta_3(y)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y\beta_3(y)}{\cos(2y)}\right) = \frac{\beta_2(y)}{\cos(2y)} + 2\left(\frac{y\beta_2(y)}{\cos(2y)}\right)^2. \tag{4.22}$$

Now integrate each side from 0 to y, restricting $|y| < \pi/4$ to avoid the poles of $\sec(2y)$. Therefore if $|y| < \pi/4$,

$$\beta_3(y) = \frac{\cos(2y)}{y}\int_0^y \frac{\beta_2(u)}{\cos(2u)} + 2\left(\frac{u\beta_2(u)}{\cos(2u)}\right)^2 \mathrm{d}u. \tag{4.23}$$

It might be of some interest to try to generalize the system of linear differential equations presented in (4.20). For example, can we perturb the coefficients, but still find a reduction to a first order system of equations? It appears (after some effort) that the equations in (4.20), combined with the initial conditions $\{\beta_0(y)=1, d_0(y)=0, \beta_k(0)=1, d_k(0)=0\}$ are not easily generalized.

Next we will establish a recursive formula for generating the Taylor series coefficients of $\nu_k(x)$ about $x=r+1/2$ where $r \in \mathbf{Z}$.

Theorem 4.7: *Let r be any integer, then we may compute the Taylor series for $\nu_k(x)$ about $x=r+1/2$. The derivatives of $\nu_k(x)$ can be computed recursively using*

$$\nu_k^{(n+1)}(r+1/2) = \sum_{m=0}^{n} 2^{m+1}(m+1)!\binom{n}{m}\frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}\left[\nu_k(x)\beta_{m+2}^{(r+1,r+1)}(\nu_k(x))\right]_{x=r+1/2}.$$

Since $J_{r+1/2}(y)$ can always be written in terms of elementary functions, $\nu_k(r+1/2)$ is always the root of an elementary function. The selection of k determines which zero of $J_x(y)$ that $\nu_k(x)$ will pass through.

TABLE I. Table of the first five positive zeros of $J_0(y)$. The approximations were calculated using Eq. (4.24).

Zero number	Exact zero	Approximate zero
1	2.4048…	2.4094…
2	5.5200…	5.5217…
3	8.6537…	8.6545…
4	11.7915…	11.7919…
5	14.9309…	14.9312…

To compute the Taylor series for $\nu_k(x)$ about $x=1/2$, apply Theorem 4.7 with $r=0$. Since $J_{1/2}(y)=(2/\pi y)^{\frac{1}{2}}\sin(y)$, the kth zero of $J_{1/2}(y)$ to the right-hand side of $x=1/2$ is just πk, therefore $\nu_k(1/2)=\pi k$. Computing the first few terms in our series, we get

$$\nu_k(x) = \pi k + 2\pi k\beta_2(\pi k)(x-1/2) - 8\pi k\beta_3(\pi k)\frac{(x-1/2)^2}{2!} + (48\pi k\beta_4(\pi k) - 64\pi^3k^3\beta_2^3(\pi k) - 24\pi k\beta_2^2(\pi k))\frac{(x-1/2)^3}{3!} + \cdots . \tag{4.24}$$

Since $\nu_k(x)$ is not analytic at $x=-1$, Eq. (4.24) converges for $-1<x<2$. We can approximate the first few zeros of $J_0(y)$ using (4.24). As Table I illustrates, this series converges very slowly.

We will pose three open problems to conclude this section.

Open question 1: It is easy to see that if $m\geq 0$ is an integer, then $\nu_k^{(m)}(1/2)$ is a polynomial in the elements $\pi k,\beta_2(\pi k),\beta_3(\pi k),\ldots,\beta_{m+1}(\pi k)$. Thus $\nu_k^{(m)}(1/2)$ can be expressed as a polynomial in πk and the derivatives of $J_x(\pi k)J_{-x}(\pi k)[\pi/\sin(\pi x)]$ at $x=1/2$. Based on this evidence, we might conjecture the existence of a "nice" function $f(x,y,z)$, such that

$$\nu_k(x) = f\left(x,\pi k,\frac{\pi}{\sin(\pi x)}J_x(\pi k)J_{-x}(\pi k)\right). \tag{4.25}$$

It would be highly desirable to prove the existence of such a function, as it would give an exact solution of $J_x(y)=0$.

Open question 2: Are the elements of the set $\{y,\cos(2y),\beta_2(y),\beta_3(y),\ldots\}$ algebraically independent? For example, can $\beta_3(y)$ be expressed in terms of $\beta_2(y)$ and trigonometric functions?

This is not a trivial question. Using Eq. (4.17) we easily see that if $n\geq 0$ is an integer,

$$\beta_3\left((2n+1)\frac{\pi}{4}\right) = (-1)^n(2n+1)\frac{\pi}{4}\beta_2^2\left((2n+1)\frac{\pi}{4}\right). \tag{4.26}$$

This shows that $\beta_3(y)$ can be expressed in terms of $\beta_2(y)$ at certain discrete points. Whether or not this indicates more general algebraic relations remains unanswered.

Open question 3: Is it possible to simplify the recurrence relation given in Theorem 4.7? In other words, can we generate $\nu_k^{(m)}(1/2)$ in a way that does not involve taking derivatives?

V. THE TAYLOR SERIES FOR $\nu_k(x)$ AT THE INTEGERS

In general, calculating the Taylor series for $\nu_k(x)$ at the integers is far more challenging than computing the Taylor series at the half-integers. Recall that $\nu_k(x)$ is discontinuous when $x\in\{-1,-2,\}$, so it only makes sense to calculate a Taylor series for $x\in\{0,1,2,\ldots\}$. The main difficulty lies in the fact that

$$\nu_k'(x) = \frac{\nu_k(x)}{2} \sum_{n=-\infty}^{\infty} \frac{J_n^2(\nu_k(x))}{(n+x)^2}$$

has removable singularities at $x=0,1,2,\ldots$. This gives rise to polynomial rather than linear relationships between derivatives of $\nu_k(x)$. To illustrate this point, we will calculate the first Taylor polynomial of $\nu_k(x)$ at zero.

Proposition 5.1: The first Taylor polynomial of $\nu_k(x)$ at zero is given by

$$\nu_k(x) = \nu_k(0) + \frac{x}{2\nu_k(0) J_1^2(\nu_k(0))} + O(x^2). \tag{5.1}$$

Proof: We simply need to show that $\nu_k'(0)$ has the correct form, and consequently Eq. (5.1) really is the first Taylor polynomial of $\nu_k(x)$ at zero.

Applying Proposition 3.2, we obtain the following relation for $\nu_k'(0)$:

$$\frac{\nu_k'(0)}{\nu_k(0)} = \frac{1}{2} \lim_{x\to 0}\left(\frac{J_0^2(\nu_k(x))}{x^2}\right) + \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{J_n^2(\nu_k(0))}{n^2} = \frac{1}{2} J_1^2(\nu_k(0))(\nu_k'(0))^2 + \sum_{n=1}^{\infty} \frac{J_n^2(\nu_k(0))}{n^2}.$$

We will use Eq. (5.4) to simplify this expression. By letting $y \to \nu_k(0)$ in (5.4), we can show that

$$\sum_{n=1}^{\infty} \frac{J_n^2(\nu_k(0))}{n^2} = \frac{1}{2\nu_k^2(0) J_1^2(\nu_k(0))}. \tag{5.2}$$

Eliminating this infinite series and solving the resulting quadratic equation for $\nu_k'(0)$, we arrive at the simple formula

$$\nu_k'(0) = \frac{1}{2\nu_k(0) J_1^2(\nu_k(0))}. \tag{5.3}$$

The fact that we can use Eq. (5.2) to prove such a simple expression for $\nu_k'(0)$ is a minor miracle. The fact that we must solve a quadratic equation for $\nu_k'(0)$ illustrates why we should not expect to find a simple recursive formula like that in Theorem 4.7 to compute $\nu_k^{(m)}(0)$. ∎

When we are computing higher derivatives of $\nu_k(x)$ at integer points, we will encounter the following three functions:

$$\mu_k(y) = \sum_{n=1}^{\infty} \frac{J_n^2(y)}{n^k},$$

$$\eta_k(y) = \sum_{n=1}^{\infty} \frac{J_{n+1}^2(y)}{n^k},$$

$$\rho_k(y) = \sum_{n=1}^{\infty} \frac{J_n(y) J_{n+1}(y)}{n^k}.$$

In Proposition 4.4 we showed that $d_k(y)$ is expressible in terms of $\beta_j(y)$'s and the elementary functions. Unfortunately such strong relationships are probably not possible between $\mu_k(y)$, $\eta_k(y)$, and $\rho_k(y)$. The strongest relationships that we have found allow us to eliminate $\rho_k(y)$ if k is even. It is not too difficult to show that

$$\rho_2(y) = -\frac{(J_0^2(y)-1)^2}{4y^2 J_0(y) J_1(y)} + \frac{\mu_1(y)}{y} - \frac{J_0(y)}{J_1(y)} \frac{\rho_1(y)}{y} + \frac{1}{2} \frac{J_1(y)}{J_0(y)} \mu_2(y) + \frac{1}{2} \frac{J_0(y)}{J_1(y)} \eta_2(y). \tag{5.4}$$

Equation (5.4) was critical in our computation of the first Taylor polynomial of $\nu_k(x)$ at $x=0$. We will state the more general case in the next proposition.

Proposition 5.2: Define a_n, b_n, and c_n *as follows:*

$$a_n = \begin{cases} J_0(y)J_1(y) & \text{if } n=-1, \\ -\dfrac{J_0^2(y)}{y} & \text{if } n=0, \\ \dfrac{2}{y}\mu_n(y) - (1-(-1)^n)\rho_{n+1}(y) & \text{if } n \geqslant 1, \end{cases}$$

$$b_n = \begin{cases} J_0^2(y) & \text{if } n=-1, \\ -(1-(-1)^n)\mu_{n+1}(y) & \text{if } n \geqslant 0, \end{cases}$$

$$c_n = \begin{cases} -J_1^2(y) & \text{if } n=-1, \\ \dfrac{2}{y}J_0(y)J_1(y) & \text{if } n=0, \\ (1-(-1)^n)\eta_{n+1}(y) + \dfrac{4}{y^2}\mu_{n-1}(y) - \dfrac{4}{y}\rho_n(y) & \text{if } n \geqslant 1. \end{cases}$$

Then for $n=0,1,2\ldots,$

$$\sum_{k=-1}^{n+1} c_k b_{n-k} = \frac{\delta_{n0}}{y^2} - \sum_{k=-1}^{n+1} a_k a_{n-k}, \tag{5.5}$$

where

$$\delta_{n0} = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise}. \end{cases}$$

If n is odd, then Eq. (5.5) is trivial. When n is even, we obtain nontrivial relationships between $\eta_j(y)$'s, $\rho_j(y)$'s, *and* $\mu_j(y)$'s. *The case where* $n=0$ *yields Eq. (5.4).*

If we let $n=2$, Eq. (5.5) yields the complicated formula

$$0 = \frac{2}{y^2}\mu_1^2(y) - 2\left(\frac{1-J_0^2(y)}{y^2}\right)\mu_2(y) + 2\frac{J_0(y)J_1(y)}{y}\mu_3(y) + J_1^2(y)\mu_4(y) - 2\mu_2(y)\eta_2(y) + J_0^2(y)\eta_4(y) + \frac{4}{y}\mu_2(y)\rho_1(y) - \frac{4}{y}\mu_1(y)\rho_2(y) + 2\rho_2^2(y) - 2\frac{J_0^2(y)}{y}\rho_3(y) - 2J_0(y)J_1(y)\rho_4(y). \tag{5.6}$$

To conclude this section, we will list a couple of interesting open questions about $\mu_k(y)$, $\eta_k(y)$, and $\rho_k(y)$.

Open question 4: Is it possible to find explicit expressions for $\mu_1(y)$, $\eta_1(y)$, and $\rho_1(y)$ in terms of known functions? It seems likely that these functions are somehow related to Bessel functions of the first and second kinds. Formulas similar to Eq. (4.2) and (4.5) would be desirable.

Open question 5: Do more nontrivial algebraic relations exist between $\mu_k(y)$, $\eta_k(y)$, and $\rho_k(y)$? This question is probably very hard, given the difficulty of working with these particular functions.

VI. A CONNECTION TO THE RIEMANN ZETA FUNCTION, AND FURTHER REMARKS ABOUT SERIES OF SQUARES OF BESSEL FUNCTIONS

The functions that we have considered so far, $\{\beta_k(y), d_k(y), \beta_k^{(a,b)}(y), \mu_k(y), \eta_k(y), \rho_k(y)\}$, are particularly nice. Besides their various algebraic relations and differentiation formulas, they are sums over products of Bessel functions of integer order. As a result, integrals like

$$\frac{1}{2} = \int_0^\infty J_n(y)J_{n+1}(y)\mathrm{d}y \quad \text{if } n \in \mathbf{Z} \text{ and } n \geqslant 0, \tag{6.1}$$

naturally connect them to the Riemann zeta function (Ref. 1, p. 1102).

Applying formula (6.1) to $d_k(y)$, it is easy to see that for $k=1,2,3,\ldots$,

$$\left(1 - \frac{1}{2^{2k+1}}\right)\zeta(2k+1) = \int_0^\infty d_{2k+1}(y)\mathrm{d}y. \tag{6.2}$$

Setting $k=1$ gives the following integral for $\zeta(3)$:

$$\frac{7}{8}\zeta(3) = \int_0^\infty d_3(y)\mathrm{d}y = \int_0^\infty (\tan(2y)\beta_3(y) - y\sec(2y)\beta_2^2(y))\mathrm{d}y.$$

A simple integration by parts generates another nontrivial integral for $\zeta(3)$. In general we can use the algebraic relations between our functions to find many complicated integrals for π and the odd values of the Riemann zeta function. Perhaps additional knowledge about the $\beta_k(y)$'s will yield some useful information about $\zeta(3), \zeta(5), \ldots$ (unfortunately this is probably too much to hope for).

Since the entire paper (up to this point) discusses functions defined by infinite series of squares of Bessel functions, we pose the following general question: What functions can be expressed in such a form?

As an example, we evaluated $\beta_2(y)$ in terms of the sine integral in formula (4.3). This expansion converges rapidly, and provides an example of a useful special function expressible by a series of squares of Bessel functions. We can also derive a similar expression for the cosine integral,

$$\int_0^{2y} \frac{1-\cos(z)}{z}\mathrm{d}z = 4\sum_{n=1}^{\infty} J_n^2(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}. \tag{6.3}$$

Proof: We can prove (6.3) by showing that the derivatives of each side of the equation are equal, and by showing that the equation holds when $y=0$.

It is trivial to show that (6.3) holds when $y=0$, since both sides of the equation vanish. Therefore we just have to show that

$$\frac{1-\cos(2y)}{y} = 4\frac{d}{\mathrm{d}y}\left(\sum_{n=1}^{\infty} J_n^2(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}\right).$$

Observe that this sum converges uniformly whenever $y \in \mathbf{R}$, so we may interchange summation and differentiation. Therefore if $y \in \mathbf{R}$,

$$\begin{aligned} 4\frac{d}{\mathrm{d}y}\left(\sum_{n=1}^{\infty} J_n^2(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}\right) &= 4\sum_{n=1}^{\infty}\frac{d}{\mathrm{d}y}(J_n^2(y))\sum_{k=0}^{n-1}\frac{1}{2k+1} \\ &= 4\sum_{n=1}^{\infty}(J_{n-1}(y)J_n(y) - J_n(y)J_{n+1}(y))\sum_{k=0}^{n-1}\frac{1}{2k+1}. \end{aligned}$$

Now break the sum into two pieces, and combine the two pieces so that the inner sums telescope,

$$=4\sum_{n=1}^{\infty}J_{n-1}(y)J_n(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}-4\sum_{n=1}^{\infty}J_n(y)J_{n+1}(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}$$

$$=4\sum_{n=0}^{\infty}J_n(y)J_{n+1}(y)\sum_{k=0}^{n}\frac{1}{2k+1}-4\sum_{n=1}^{\infty}J_n(y)J_{n+1}(y)\sum_{k=0}^{n-1}\frac{1}{2k+1}$$

$$=4\sum_{n=0}^{\infty}J_n(y)J_{n+1}(y)\left(\sum_{k=0}^{n}\frac{1}{2k+1}-\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)=4\sum_{n=0}^{\infty}\frac{J_n(y)J_{n+1}(y)}{2n+1}.$$

Using the fact that $J_{-n}(y)=(-1)^n J_n(y)$ [as in the proof of Eq. (2.4)], this becomes

$$2\sum_{n=-\infty}^{\infty}\frac{J_n(y)J_{n+1}(y)}{2n+1}=\frac{1-\cos(2y)}{y}$$

by Eq. (4.5). ■

In the next theorem we state Gegenbauer's result from the 1870s, namely that any even function with a Taylor series at zero can be written as a sum of squares of Bessel functions (Ref. 5, p. 525). The form that we give for this theorem is slightly different from Watson's statement, but is sufficient for our purposes.

Theorem 6.1: *(Gegenbauer) Suppose that $f(z)$ is an even function that is analytic at zero, or equivalently suppose that $f(z)$ is even and has a Taylor series at zero. Let*

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(2n)}(0)}{(2n)!}z^{2n}.$$

Then the following representation holds for $f(z)$:

$$f(z)=\sum_{n=0}^{\infty}a_n J_n^2(z),$$

where a_n is given by

$$a_n=\begin{cases} f(0) & \text{if } n=0,\\ 2n\sum_{k=0}^{n}\frac{\binom{n+k}{2k}}{n+k}\frac{2^{2k}}{\binom{2k}{k}}f^{(2k)}(0) & \text{if } n\geqslant 1.\end{cases}$$

This formula relating a_n and $f^{(2n)}(0)$ can be inverted to give

$$f^{(2n)}(0)=\frac{(-1)^n}{2^{2n}}\binom{2n}{n}\sum_{k=0}^{n}(-1)^k a_k\binom{2n}{n+k}.$$

If we apply Theorem 6.1 to appropriately chosen hypergeometric functions, we can obtain a variety of "nice" series of squares of Bessel functions. Examples include

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!^3}\frac{\left(\frac{y}{2}\right)^{2n}}{(x)_n}=J_0^2(y)+2\sum_{n=1}^{\infty}(-1)^n\frac{(1-x)_n}{(x)_n}J_n^2(y), \tag{6.4}$$

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\binom{2n-x}{n-x}\frac{\left(\frac{y}{2}\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(x)_n}{n!}J_n^2(y), \tag{6.5}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$.

Open question 6: An interesting question associated with Theorem 6.1 is to find a "nice" expression for the Taylor series coefficients of $\mu_k(y)$ when k is an odd integer. It is easy to show that the Taylor series coefficients of $\beta_k(y)$, $d_k(y)$, and $\mu_{2k}(y)$ can always be expressed in terms of harmonic numbers. If we let

$$\lambda_n^{(j)}=\sum_{m=0}^{n}\frac{1}{(2m+1)^j}, \quad H_n^{(j)}=\sum_{m=1}^{n}\frac{1}{m^j},$$

then we can obtain series expansions including

$$\beta_3(y)=\sum_{n=0}^{\infty}(-1)^n\lambda_n^{(2)}\frac{(2y)^{2n}}{(2n+1)!}, \tag{6.6}$$

$$\mu_4(y)=\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{H_n^{(4)}+(H_n^{(2)})^2}{4}\right)\binom{2n}{n}\frac{\left(\frac{y}{2}\right)^{2n}}{n!^2}, \tag{6.7}$$

$$d_5(y)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{\lambda_n^{(4)}+(\lambda_n^{(2)})^2}{2}\right)\frac{(2y)^{2n}}{(2n+2)!}. \tag{6.8}$$

A simple application of Theorem 6.1 shows that trying to find nice Taylor series coefficients for $\{\mu_3(y),\mu_5(y),\ldots\}$ is equivalent to reducing

$$r_n^{(j)}=\sum_{m=1}^{n}\frac{(-1)^{m+1}}{m^j}\binom{2n}{n+m},$$

into "some nice form" when $j>1$ is an odd integer.

In the case of $\mu_1(y)$, we can show that if A_n is the alternating harmonic series,

$$A_n=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k},$$

then we must have

$$\mu_1(y)=\sum_{n=1}^{\infty}(-1)^{n+1}A_{2n}\binom{2n}{n}\frac{\left(\frac{y}{2}\right)^{2n}}{n!^2}. \tag{6.9}$$

Generalizing this result has proven surprisingly difficult.

VII. MORE REPRESENTATIONS OF $\beta_k(y)$ AND $d_k(y)$ IN TERMS OF INFINITE SERIES

The functions $\beta_k(y)$ and $d_k(y)$ possess many representations in terms of infinite integrals and infinite series. Since $\beta_k(y)$ and $d_k(y)$ are just the derivatives with respect to order of products of Bessel functions, every one of the legions of representations for Bessel functions will yield formulas for $\beta_k(y)$ and $d_k(y)$. In this section we will present a few of the most visually appealing and useful formulas that we have encountered.

Our first formula expresses $d_k(y)$ in terms of sums running over the Bessel functions of half-integer order. This is noteworthy since the Bessel functions of half-integer order reduce to polynomials in $1/\pi$ whenever $y\to\pi k$.

Let $\alpha_n(y)$ be defined by

$$\alpha_n(y)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^n}J_{1/2+k}(y)J_{1/2-k}(y), \tag{7.1}$$

and let B_j denote the Bernoulli numbers (Ref. 1, p. 1107). Then for $n\geqslant 1$,

$$d_{2n-1}(y)=\frac{4(4^n-1)}{(2n)!}|B_{2n}|\pi^{2n-2}\frac{\sin^2(y)}{y}+\frac{1}{4^{n-1}}\sum_{j=1}^{n-1}\frac{4^j(4^j-1)}{(2j)!}|B_{2j}|\pi^{2j-1}\alpha_{2n-2j}(y). \tag{7.2}$$

Equation (7.2) yields the following relation when $n=2$:

$$d_3(y)=\frac{\pi^2}{12}\frac{\sin^2(y)}{y}+\frac{\pi}{4}\alpha_2(y). \tag{7.3}$$

A second family of formulas can be obtained from Eqs. (2.9) and (2.10). We can decompose $\beta_k(y)$ and $d_k(y)$ in terms of two sets of functions defined by a modified Neumann series. Define $g_k(y)$ and $h_k(y)$ by

$$g_k(y)=\sum_{n=0}^{\infty}\frac{J_n(y)\left(\frac{y}{2}\right)^n}{n!(2n+1)^k}, \tag{7.4}$$

$$h_k(y)=\sum_{n=0}^{\infty}\frac{J_{n+1}(y)\left(\frac{y}{2}\right)^n}{n!(2n+1)^k}. \tag{7.5}$$

It is easy to see from Eq. (2.2) that $g_k(y)$ and $h_k(y)$ can be written compactly as

$$g_k(y)=\frac{(-1)^{k-1}}{2^k(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left[\left(\frac{2}{y}\right)^x\Gamma(x)J_x(y)\right]_{x=1/2}\quad \text{if } k\geqslant 1, \tag{7.6}$$

$$h_k(y)=\frac{-1}{2^k(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left[\left(\frac{2}{y}\right)^{1-x}\Gamma(1-x)J_{-x}(y)\right]_{x=1/2}\quad \text{if } k\geqslant 2. \tag{7.7}$$

We can prove the following formulas for $\beta_k(y)$ and $d_k(y)$:

$$d_{2n+1}(y)=2h_{2n+1}(y)-y\sum_{m=0}^{2n}(-1)^m h_{m+1}(y)h_{2n+1-m}(y), \tag{7.8}$$

$$d_{2n+1}(y)=y\sum_{m=0}^{2n}(-1)^m g_{m+1}(y)g_{2n+1-m}(y), \tag{7.9}$$

$$\beta_n(y)=g_n(y)-y\sum_{m=0}^{n}(-1)^m h_{m+1}(y)g_{n-m}(y). \tag{7.10}$$

By combining these three formulas we can find a wide variety of relations between $\beta_k(y)$, $d_k(y)$, $h_k(y)$, and $g_k(y)$. We will use the following elementary evaluations of $g_0(y)$, $g_1(y)$, $h_0(y)$, and $h_1(y)$,

$$g_0(y) = 1, \quad g_1(y) = \frac{\sin(y)}{y}, \tag{7.11}$$

$$h_0(y) = \frac{y}{2}, \quad h_1(y) = \frac{1-\cos(y)}{y}. \tag{7.12}$$

Examples of identities include

$$h_2(y)\sin(y) + g_2(y)\cos(y) = \beta_2(y), \tag{7.13}$$

$$h_3(y)\sin(y) + g_3(y)\cos(y) = \frac{1}{\sin(2y)}(d_3(y) + 2y\cos(y)g_2(y)\beta_2(y) - y\beta_2^2(y)), \tag{7.14}$$

$$h_4(y)\sin(y) + g_4(y)\cos(y) = \beta_4(y) + y(g_2(y)h_3(y) - h_2(y)g_3(y)). \tag{7.15}$$

Unfortunately, it seems to be impossible to express $g_k(y)$ or $h_k(y)$ only in terms of $\beta_k(y)$'s. This is not especially surprising, since we would not intuitively expect to find strong relationships between the derivatives of $J_x(y)$ and the derivatives of $J_x(y)J_{-x}(y)$. Interestingly enough however, it may be possible to express $g_k(y)$ in terms of $\beta_k(y)$'s at certain arguments. An example follows easily from Eq. (7.13),

$$g_2(\pi k) = (-1)^k\beta_2(\pi k) \quad \text{for } k \in \mathbf{Z}. \tag{7.16}$$

A third class of identities can be derived from Eqs. (2.12) and (2.13). While we have not explored this avenue in depth, we have derived a few noteworthy formulas. If we let

$$F_k(y,z) = \sum_{n=-\infty}^{\infty} \frac{J_n(y)J_n(z)}{(2n+1)^k}\left(\frac{y}{z}\right)^n,$$

$$G_k(y,z) = \sum_{n=-\infty}^{\infty} \frac{J_n(y)J_{n+1}(z)}{(2n+1)^k}\left(\frac{y}{z}\right)^n,$$

then examples of identities include

$$d_3(y) = -\left(\frac{\pi}{2}\right)^2\beta_2^2\left(\frac{\pi}{2}\right)\frac{\sin^2(y)}{y} - yG_2^2\left(y,\frac{\pi}{2}\right) + 2\sin(y)G_3\left(y,\frac{\pi}{2}\right), \tag{7.17}$$

$$\frac{\sin(2z)}{2z}\beta_2(y) + \frac{\sin(2y)}{2y}\beta_2(z) = \frac{\sin(z)}{z}\cos(y)F_2(y,z) + \frac{\sin(y)}{y}\cos(z)F_2(z,y). \tag{7.18}$$

VIII. CONCLUSION

It would be particularly nice to see solutions to some of the open problems presented in this paper. A solution of question 4 could potentially yield a rapidly converging series for $Y_0(z)$, while a solution for question 3 would simplify Theorem 4.7. Such results would be quite useful.

It might also be interesting to generalize the results of Secs. II, IV, and V to other special functions. For example, while it is possible to generate a differential equation similar to Eq. (3.2) for the zeros of ${}_2F_1\left(\begin{matrix}a,1-a\\c\end{matrix}\middle|x\right)$, it is not immediately clear that this will yield a nice series similar to Eq. (4.24) for the zeros of this hypergeometric function.

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[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products* (Academic, New york, 1994).
[2] Lommel, E. C., *Studien uber die Bessel'schen Functionen*, Leipzig, 1868.
[3] Muldoon, M. E., "Convexity properties of special functions and their zeros," *Recent Progress in Inequalities* (Kluwer Academic, New York 1997).
[4] Schafheitlin, V. P., *Die Theorie der Besselschen Funktionen*, Leipzig and Berlin, 1908.
[5] Watson, G. N., *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922).