

# TWO-DIMENSIONAL SERIES EVALUATIONS VIA THE ELLIPTIC FUNCTIONS OF RAMANUJAN AND JACOBI

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**Abstract.** We evaluate in closed form, for the first time, certain classes of double series, which are remindful of lattice sums. Elliptic functions, singular moduli, class invariants, and the Rogers–Ramanujan continued fraction play central roles in our evaluations.

## 1. INTRODUCTION

In this paper we establish elementary evaluations of certain 2-dimensional infinite series. For example,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(5m)^2 + (5n+1)^2} = -\frac{\pi}{5\sqrt{5}} \log \left( \sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}} \right) + \frac{\pi}{25} \log \left( 11 + 5\sqrt{5} \right), \quad (1.1)$$

which is a problem submitted to the *American Mathematical Monthly* [15]. The algebraic numbers on the right-hand side of (1.1) arise from special values of the Rogers–Ramanujan continued fraction. In general, elementary evaluations are quite rare for higher-dimensional lattice-type sums. For instance, the third author has examined both double and quadruple sums in connection with Mahler measures of elliptic curves; those sums typically reduce to values of hypergeometric functions [10], [11], [12]. The most famous higher-dimensional sum is the Madelung constant from crystallography [5], [7], [8], [14]. It is highly unlikely that Madelung’s constant possesses an evaluation in closed form.

We produce many additional results along the lines of (1.1). In fact, we show that it is possible to evaluate

$$F_{(a,b)}(x) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2} \quad (1.2)$$

for any positive rational value of  $x$ , and for many values of  $(a, b) \in \mathbb{N}^2$ . Since the series is not absolutely convergent, we will calculate the  $n$ -index of summation using  $\sum_n = \lim_{N \rightarrow \infty} \sum_{-N < n < N}$ . When  $a \leq 6$ , results from Ramanujan’s notebooks apply, and the values of the sums can be deduced from classical results in theta functions and  $q$ -series. When  $a > 6$ , the situation is slightly more complicated. In those cases we require the added hypothesis that  $a \in 2\mathbb{Z}$ , and then we use properties of Jacobian elliptic functions.

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## 2. THE MAIN THEOREM

Before proving our main theorem, we note that it is possible to calculate  $F_{(a,b)}(x)$  to high numerical precision with the formula

$$F_{(a,b)}(x) = \frac{\pi}{x} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \operatorname{csch}\left(\frac{\pi(an+b)}{x}\right)}{an+b}, \quad (2.1)$$

which can be established by substituting the partial fractions decomposition for  $\operatorname{csch}(z)$  [9, p. 28, Entry 1.217] in (1.2). Formula (2.1) provides an easy way to numerically verify results such as (1.1) and (3.20). For instance, we calculated  $F_{(5,1)}(1)$  to more than 100 decimal places by summing over  $-15 \leq n \leq 15$ .

**Theorem 2.1.** *Suppose that  $a$  and  $b$  are integers with  $a \geq 2$ ,  $(a, b) = 1$ , and assume that  $\operatorname{Re}(x) > 0$ . Then*

$$F_{(a,b)}(x) = -\frac{2\pi}{ax} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \left( \prod_{m=0}^{\infty} (1 - \omega^{2j+1} q^{2m+1}) (1 - \omega^{-2j-1} q^{2m+1}) \right), \quad (2.2)$$

where  $\omega = e^{\pi i/a}$  and  $q = e^{-\pi/x}$ .

*Proof.* Suppose that  $N$  is a positive integer. Then using the transformation formula for the theta function  $\varphi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$  [3, p. 43, Entry 27(ii)] and inverting the order of summation and integration twice by absolute convergence, we find that

$$\begin{aligned} \sum_{\substack{m \in \mathbb{Z} \\ -N < n < N}} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2} &= \pi \sum_{-N < n < N} (-1)^n \int_0^{\infty} e^{-\pi(an+b)^2 u} \left( \sum_{m \in \mathbb{Z}} (-1)^m e^{-\pi m^2 x^2 u} \right) du \\ &= \pi \sum_{-N < n < N} (-1)^n \int_0^{\infty} e^{-\pi(an+b)^2 u} \left( \frac{2}{x\sqrt{u}} \sum_{m=0}^{\infty} e^{-\frac{\pi(2m+1)^2}{4x^2 u}} \right) du \\ &= \frac{2\pi}{x} \sum_{m=0}^{\infty} \sum_{-N < n < N} (-1)^n \int_0^{\infty} e^{-\pi(an+b)^2 u} e^{-\frac{\pi(2m+1)^2}{4x^2 u}} \frac{du}{\sqrt{u}}. \end{aligned}$$

The substitution of a standard  $K$ -Bessel function integral [9, p. 384, formula 3.471, no. 9]

$$\int_0^{\infty} e^{-\pi(A^2 u + B^2/u)} \frac{du}{\sqrt{u}} = \frac{e^{-2\pi|A||B|}}{|A|}$$

leads to

$$\sum_{\substack{m \in \mathbb{Z} \\ -N < n < N}} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2} = \frac{2\pi}{x} \sum_{m=0}^{\infty} \sum_{-N < n < N} \frac{(-1)^n q^{(2m+1)|an+b|}}{|an+b|}. \quad (2.3)$$

Notice that the condition  $\operatorname{Re}(x) > 0$  ensures convergence. Now recall that  $\omega = e^{\pi i/a}$ . If  $a \geq 2$ ,  $(a, b) = 1$ , and  $|r| < 1$ , then

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n r^{|an+b|}}{|an+b|} = -\frac{1}{a} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \left( (1 - \omega^{2j+1} r) (1 - \omega^{-(2j+1)} r) \right), \quad (2.4)$$

which is easily verified by comparing Taylor series coefficients in  $r$ . Notice that (2.4) involves an infinite series in  $n$ , whereas (2.3) imposes the restriction that  $n \in (-N, N)$ . If we divide the left-hand side of (2.4) into three components,  $-\infty < n \leq -N$ ,  $-N < n < N$ ,  $N \leq n < \infty$ , and use a crude error estimate to bound the terms where  $n \geq N$  and  $n \leq -N$ , then we find that

$$\sum_{-N < n < N} \frac{(-1)^n r^{|an+b|}}{|an+b|} = -\frac{1}{a} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log((1 - \omega^{2j+1}r)(1 - \omega^{-(2j+1)}r)) + O\left(\frac{r^N}{(1-r)N}\right). \quad (2.5)$$

Substituting (2.5) into (2.3) leads to equation (2.2) plus an error term. The error term can easily be seen to approach zero as  $N \rightarrow \infty$ .  $\square$

### 3. SIMPLIFICATION FOR $a \in \{3, 4, 5, 6\}$

Although (2.2) was not difficult to prove, the deduction of results such as (1.1) from (2.2) is usually more difficult. In this section we examine the cases where  $a \in \{3, 4, 5, 6\}$ . In these instances we can assume that  $b = 1$  without loss of generality, because a standard symmetry (e.g.  $n \rightarrow -n$ ) can be used to recover the other possible values of  $F_{(a,b)}(x)$ . The same symmetry immediately implies that  $F_{(2,1)}(x) = 0$ .

Let us briefly recall the  $q$ -series notation

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j), \quad |q| < 1.$$

Following Ramanujan's notation for theta functions, define

$$\begin{aligned} \varphi(q) &= \sum_{n=-\infty}^{\infty} q^{n^2}, & \psi(q) &= \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \\ \chi(q) &= (-q; q^2)_\infty, & f(-q) &= (q; q)_\infty. \end{aligned}$$

We need the famous Jacobi triple product identity [3, p. 35, Entry 19] for Ramanujan's general theta function  $f(a, b)$ , and they are given by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1. \quad (3.1)$$

We note the easily proved evaluation [3, p. 34, Entry 18(ii)]

$$f(-1, q) = 0, \quad |q| < 1. \quad (3.2)$$

The Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \quad |q| < 1,$$

also plays an important role in this paper. We always assume that  $q^{1/5}$  takes the principal value, so that  $R(-q)$  assumes a real value if  $q \in (0, 1)$ .

**Theorem 3.1.** *Suppose that  $q = e^{-\pi/x}$ . Let  $u = -q^{1/3}\chi(q)/\chi^3(q^3)$ ,  $\alpha = 1 - \varphi^4(-q^8)/\varphi^4(q^8)$ , and  $\mu = R(-q)R^2(q^2)$ . Then*

$$F_{(3,1)}(x) = -\frac{2\pi}{9x} \log \left( \frac{1 + u^3}{1 - 8u^3} \right), \quad (3.3)$$

$$F_{(4,1)}(x) = -\frac{\pi}{\sqrt{2}x} \log \left( \frac{1 - \sqrt[8]{\alpha}}{1 + \sqrt[8]{\alpha}} \right), \quad (3.4)$$

$$F_{(5,1)}(x) = -\frac{\pi}{5\sqrt{5}x} \log \left( \frac{2 - \mu + 18\mu^2 + \mu^3 + 2\mu^4 + 5\sqrt{5}(\mu + \mu^3)}{2 - \mu + 18\mu^2 + \mu^3 + 2\mu^4 - 5\sqrt{5}(\mu + \mu^3)} \right) \\ - \frac{\pi}{5x} \log \left( \frac{1 + \mu - \mu^2}{1 - 4\mu - \mu^2} \right), \quad (3.5)$$

$$F_{(6,1)}(x) = -\frac{\pi}{\sqrt{3}x} \log \left( \frac{\varphi(-q^4) - 3\varphi(-q^{36}) + 2\sqrt{3}qf(-q^{24})}{\varphi(-q^4) - 3\varphi(-q^{36}) - 2\sqrt{3}qf(-q^{24})} \right). \quad (3.6)$$

*Proof.* We begin by proving (3.3). If we set  $(a, b) = (3, 1)$ , then (2.2) immediately reduces to

$$F_{(3,1)}(x) = -\frac{2\pi}{3x} \sum_{j=0}^2 \cos \left( \frac{\pi(2j+1)}{3} \right) \log \prod_{m=0}^{\infty} \left( 1 - 2 \cos \left( \frac{\pi(2j+1)}{3} \right) q^{2m+1} + q^{4m+2} \right) \\ = -\frac{2\pi}{3x} \log \prod_{m=0}^{\infty} \frac{1 - q^{2m+1} + q^{4m+2}}{1 + 2q^{2m+1} + q^{4m+2}} = -\frac{2\pi}{3x} \log \frac{\chi(q^3)}{\chi^3(q)}. \quad (3.7)$$

To finish the calculation, let us briefly take  $\alpha = 1 - \varphi^4(-q)/\varphi^4(q)$  and  $\beta = 1 - \varphi^4(-q^3)/\varphi^4(q^3)$ . Then by the inversion formula for  $q^{-1/24}\chi(q)$  [3, p. 124, Entry 12(v)], we have

$$\frac{\chi^3(q)}{\chi^3(q^3)} = 2^{1/3} \left( \frac{\beta(1-\beta)}{\alpha^3(1-\alpha)^3} \right)^{1/24}, \quad \frac{\chi^3(q^3)}{q^{1/3}\chi(q)} = 2^{1/3} \left( \frac{\alpha(1-\alpha)}{\beta^3(1-\beta)^3} \right)^{1/24}.$$

It is known that  $\alpha$  and  $\beta$  admit birational parametrizations  $\alpha = p(2+p)^3/(1+2p)^3$  and  $\beta = p^3(2+p)/(1+2p)$  [3, p. 230, Entry 5(vi)]. Thus we obtain

$$\frac{\chi(q^3)}{\chi^3(q)} = \left( \frac{(1-p)(2+p)}{2(1+2p)^2} \right)^{1/3}, \quad \frac{q^{1/3}\chi(q)}{\chi^3(q^3)} = \left( \frac{p(1+p)}{2} \right)^{1/3}.$$

Finally, it is easy to see that if  $u := -q^{1/3}\chi(q)/\chi^3(q^3)$ , then

$$\frac{\chi(q^3)}{\chi^3(q)} = \left( \frac{1 + u^3}{1 - 8u^3} \right)^{1/3}.$$

Substituting this last result into (3.7) completes the proof of (3.3).

Next we prove (3.4). Notice that if  $(a, b) = (4, 1)$ , then (2.2) becomes

$$\begin{aligned} F_{(4,1)}(x) &= -\frac{2\pi}{4x} \sum_{j=0}^3 \cos\left(\frac{\pi(2j+1)}{4}\right) \log \prod_{m=0}^{\infty} \left(1 - 2 \cos\left(\frac{\pi(2j+1)}{4}\right) q^{2m+1} + q^{4m+2}\right) \\ &= -\frac{\pi}{\sqrt{2}x} \log \prod_{m=0}^{\infty} \frac{(1 - \sqrt{2}q^{2m+1} + q^{4m+2})(1 - q^{2m+2})}{(1 + \sqrt{2}q^{2m+1} + q^{4m+2})(1 - q^{2m+2})}. \end{aligned} \quad (3.8)$$

Letting  $\omega = e^{\pi i/4}$  and using the Jacobi triple product identity (3.1), we find that the denominator on the far right side of (3.8) is equal to

$$\begin{aligned} F(q) &:= \prod_{m=0}^{\infty} \left(1 + \sqrt{2}q^{2m+1} + q^{4m+2}\right) (1 - q^{2m+2}) \\ &= (-\omega q; q^2)_{\infty} (-\bar{\omega} q; q^2)_{\infty} (q^2; q^2)_{\infty} \\ &= \sum_{m=-\infty}^{\infty} \omega^n q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{(4n)^2} + \omega q^{(4n+1)^2} + i q^{(4n+2)^2} + \omega^3 q^{(4n+3)^2}\right). \end{aligned} \quad (3.9)$$

Because the initial infinite product is real-valued, the imaginary terms above sum to 0. Alternatively, this fact also follows from (3.2). Hence, from (3.9),

$$\begin{aligned} F(q) &= \varphi(-q^{16}) + \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{(4n+1)^2} - q^{(4n+3)^2}\right) \\ &= \varphi(-q^{16}) + \sqrt{2} \sum_{n=0}^{\infty} (-1)^{\frac{n(n+1)}{2}} q^{(2n+1)^2} \\ &= \varphi(-q^{16}) + \sqrt{2} q \psi(-q^8). \end{aligned} \quad (3.10)$$

A similar argument provides a similar representation for the numerator on the far right side of (3.8). Hence, using (3.10) and its aforementioned analogue in (3.8), we are led to the closed form

$$F_{(4,1)}(x) = -\frac{\pi}{\sqrt{2}x} \log \frac{\varphi(-q^{16}) - \sqrt{2}q\psi(-q^8)}{\varphi(-q^{16}) + \sqrt{2}q\psi(-q^8)}.$$

If we take  $\alpha = 1 - \varphi^4(-q^8)/\varphi^4(q^8)$  and  $z = \varphi^2(q^8)$ , this expression reduces to (3.4) after applying [3, p. 122, Entry 10(iii)] and [3, p. 123, Entry 11(ii)].

Next we prove (3.5). This case is substantially more difficult than the previous two. If  $(a, b) = (5, 1)$ , then (2.2) becomes

$$\begin{aligned} F_{(5,1)}(x) &= -\frac{2\pi}{5x} \sum_{j=0}^4 \cos\left(\frac{\pi(2j+1)}{5}\right) \log \prod_{m=0}^{\infty} \left(1 - 2 \cos\left(\frac{\pi(2j+1)}{5}\right) q^{2m+1} + q^{4m+2}\right) \\ &= -\frac{\pi}{5x} \log \frac{\chi(q^5)}{\chi^5(q)} - \frac{\pi}{\sqrt{5}x} \log \prod_{m \text{ odd}}^{\infty} \frac{1 - \beta q^m + q^{2m}}{1 - \alpha q^m + q^{2m}}, \end{aligned} \quad (3.11)$$

where  $\beta = \frac{1+\sqrt{5}}{2}$  and  $\alpha = \frac{1-\sqrt{5}}{2}$ . The second equality in (3.11) was obtained by collecting terms of the form  $\sqrt{5} \log(X)$ . Now we use several entries from Ramanujan's lost notebook. By [2, pp. 21–22, Entry 1.4.1, Eqs. (1.4.3), (1.4.4)],

$$\prod_{m \text{ odd}} \left( \frac{1 - \beta q^m + q^{2m}}{1 - \alpha q^m + q^{2m}} \right) = \sqrt[5]{\frac{(1 - \alpha^5 R^5(-q))(1 - \beta^5 R^5(q^2))}{(1 - \beta^5 R^5(-q))(1 - \alpha^5 R^5(q^2))}}. \quad (3.12)$$

By [2, p. 33], we can parameterize  $R^5(-q)$  and  $R^5(q^2)$  in terms of  $\mu = R(-q)R^2(q^2)$  with the identities

$$R^5(-q) = \mu \left( \frac{1 - \mu}{1 + \mu} \right)^2, \quad R^5(q^2) = \mu^2 \left( \frac{1 + \mu}{1 - \mu} \right). \quad (3.13)$$

Therefore (3.12) becomes

$$\prod_{m \text{ odd}} \left( \frac{1 - \beta q^m + q^{2m}}{1 - \alpha q^m + q^{2m}} \right) = \sqrt[5]{\frac{2 - \mu + 18\mu^2 + \mu^3 + 2\mu^4 + 5\sqrt{5}(\mu + \mu^3)}{2 - \mu + 18\mu^2 + \mu^3 + 2\mu^4 - 5\sqrt{5}(\mu + \mu^3)}}. \quad (3.14)$$

Finally, notice that  $1/\chi(-q) = (-q; q)_\infty$ . After replacing  $q$  by  $-q$  in [2, p. 37, Entry 1.8.5] and simplifying, we have

$$\frac{\chi(q^5)}{\chi^5(q)} = \frac{1 + \mu - \mu^2}{1 - 4\mu - \mu^2}. \quad (3.15)$$

Substituting (3.14) and (3.15) into (3.11) concludes the proof of (3.5).

The proof of (3.6) is similar to the proof of (3.4), and we leave this calculation as an exercise for the reader. Note that the operative result

$$\prod_{n=0}^{\infty} \left( \frac{1 + \sqrt{3}q^{2n+1} + q^{4n+2}}{1 - \sqrt{3}q^{2n+1} + q^{4n+2}} \right) = \frac{-\varphi(-q^4) + 3\varphi(-q^{36}) + 2\sqrt{3}qf(-q^{24})}{-\varphi(-q^4) + 3\varphi(-q^{36}) - 2\sqrt{3}qf(-q^{24})}$$

follows easily from the Jacobi triple product identity (3.1). □

Now we derive some explicit examples from Theorem 3.1. All of our identities follow from well-known  $q$ -series evaluations. We begin with  $F_{(3,1)}(x)$ . Notice by [2, p. 95, Eq. 3.3.6], that  $u = G(-q)$ , where  $G(q)$  denotes Ramanujan's cubic continued fraction defined by

$$G(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \dots, \quad |q| < 1.$$

It follows that  $F_{(3,1)}(x)$  can be evaluated by using formulas for  $G(-q)$ . When  $x = 1$  we appeal to [2, p. 100, Eq. (3.4.1)] to find that  $u = G(-e^{-\pi}) = \frac{1-\sqrt{3}}{2}$ , which yields

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{m^2 + (3n+1)^2} = \frac{2\pi}{9} \log \left( 2(\sqrt{3}-1) \right). \quad (3.16)$$

When  $x = \frac{1}{\sqrt{5}}$  we appeal to [2, p. 101, Theorem 3.4.2]. We have  $u = G(-e^{-\pi\sqrt{5}}) = \frac{(\sqrt{5}-3)(\sqrt{5}-\sqrt{3})}{4}$ , and therefore

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{m^2 + 5(3n+1)^2} = \frac{\pi}{9\sqrt{5}} \log \left( 8(4 - \sqrt{15}) \right). \quad (3.17)$$

It is possible to obtain many additional formulas for  $F_{(3,1)}(x)$ , by applying formulas in [2, pp. 100–105].

Now we examine  $F_{(4,1)}(x)$ . This function is easy to examine, because when  $q = \exp(-\pi\sqrt{n})$ , where  $n \in \mathbb{Q}^+$ , the values of  $\alpha_n$  are called singular moduli and can always be calculated in terms of algebraic numbers [6, p. 214]. Their values have been extensively tabulated. For example, many explicit evaluations of  $\alpha_n$  can be found in [4, pp. 281–306]. Since  $\alpha$  has an argument of  $q^8 = e^{-8\pi/x}$ , we write  $\alpha = \alpha_{64/x^2}$ . When  $x = 8$ , then  $\alpha = \alpha_1 = 1/2$ , and therefore

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(8m)^2 + (4n+1)^2} = \frac{\pi}{8\sqrt{2}} \log \left( \frac{\sqrt[8]{2} + 1}{\sqrt[8]{2} - 1} \right). \quad (3.18)$$

Similarly, when  $x = 4/\sqrt{7}$ , we have [4, p. 284]  $\alpha = \alpha_{28} = (\sqrt{2} - 1)^8 (2\sqrt{2} - \sqrt{7})^4$  [3, p. 284]. Thus we obtain

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(4m)^2 + 7(4n+1)^2} = \frac{\pi}{4\sqrt{14}} \log \left( \frac{1 + (\sqrt{2} - 1)\sqrt{2\sqrt{2} - \sqrt{7}}}{1 - (\sqrt{2} - 1)\sqrt{2\sqrt{2} - \sqrt{7}}} \right). \quad (3.19)$$

There are many similar identities that follow from results in [4], but they often tend to be very complicated. The majority of the identities contain algebraic numbers involving nested radicals.

We conclude this section by proving a pair of formulas for  $F_{(5,1)}(x)$ . By (3.5), this requires calculating the parameter  $\mu$ . In principle, these calculations are straight-forward exercises. If the values of both  $R(-q)$  and  $R(q^2)$  are known, then calculating  $\mu = R(-q)R^2(q^2)$  is trivial. If only one of the values is known, then  $\mu$  can be calculated by solving (3.13). This second type of calculation requires solving a cubic equation. If neither value is known, then we can use (3.15) to calculate  $\mu$ . In practice, we have only been able to identify two instances where  $\mu$  is reasonably simple.

We begin by setting  $x = 1$  in (3.5). By [2, pp. 57–58],

$$\mu = R(-e^{-\pi})R^2(e^{-2\pi}) = \frac{1}{8}(3 - \sqrt{5})(7 + 3\sqrt{5}) \left( 4 + 2\sqrt{5} - \sqrt{10(5 + \sqrt{5})} \right).$$

Substituting this last result into (3.5) and simplifying with *Mathematica* leads to

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{m^2 + (5n+1)^2} \\ = -\frac{\pi}{5\sqrt{5}} \log \left( -19 + 9\sqrt{5} - 3\sqrt{85 - 38\sqrt{5}} \right) + \frac{\pi}{5} \log \left( \sqrt{5} - 1 \right). \end{aligned} \quad (3.20)$$

Now we examine the more difficult case when  $x = 5$ . This choice of  $x$  leads to (1.1) quoted in our Introduction. We calculate  $\mu$  using (3.15) and the values of class invariants

$G_n$ , which are defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q),$$

where  $q = \exp(-\pi\sqrt{n})$ . Hence, using the fact that  $G_n = G_{1/n}$  and the value of  $G_{25}$  [4, p. 190], we find that

$$\frac{1 + \mu - \mu^2}{1 - 4\mu - \mu^2} = \frac{\chi(e^{-\pi})}{\chi^5(e^{-\pi/5})} = \frac{G_1}{2G_{\frac{1}{25}}^5} = \frac{G_1}{2G_{25}^5} = \frac{1}{2} \left( \frac{\sqrt{5} - 1}{2} \right)^5.$$

Therefore  $\mu$  is given by

$$\mu = \frac{1}{4} (3 + \sqrt{5}) \left( -4 + 2\sqrt{5} - \sqrt{2(25 - 11\sqrt{5})} \right).$$

Substituting this last result into (3.5), and then simplifying nested radicals, we complete the proof of (1.1).

#### 4. SIMPLIFICATION FOR HIGHER VALUES

In this section, we evaluate  $F_{(a,b)}(x)$  when  $a > 6$ . In order to simplify the calculations, we restrict our attention to cases where  $a \in 2\mathbb{Z}$ . In these cases we can apply elementary properties of Jacobian elliptic functions. Let us briefly recall that the elliptic functions  $\operatorname{sn}(u)$ ,  $\operatorname{cn}(u)$ , and  $\operatorname{dn}(u)$  are doubly-periodic, meromorphic functions, which depend implicitly on a parameter  $\alpha = k^2$ , where  $k$  is called the elliptic modulus. Their periods are integral multiples of  $K$  and  $iK'$ , where  $K$  and  $K'$  are complete elliptic integrals of the first kind associated with the moduli  $k$  and  $k' = \sqrt{1 - k^2}$ , respectively. For us, the representations in terms of hypergeometric functions  ${}_2F_1$  [3, p. 102]

$$K := \frac{\pi}{2} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right), \quad K' := \frac{\pi}{2} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha \right)$$

are employed in the sequel. There is a well-known inverse relation between the modulus, and the elliptic nome  $q$  given by [3, p. 102]

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad q = \exp \left( -\pi \frac{{}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha \right)}{{}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)} \right).$$

**Theorem 4.1.** *Suppose that  $\alpha \in (0, 1)$ , and let*

$$x = \frac{{}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)}{{}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha \right)}. \quad (4.1)$$

*Assume that  $a \in \{2, 4, 6, \dots\}$ ,  $b \in \mathbb{Z}$ , and  $(a, b) = 1$ . Then*

$$F_{(a,b)}(x) = \frac{\pi}{ax} \sum_{j=0}^{a-1} \cos \left( \frac{\pi(2j+1)b}{a} \right) \log \left( \operatorname{dn} \left( \frac{(2j+1)K}{a} \right) \right). \quad (4.2)$$



*Proof.* We have already established that (2.2) is true if  $x \in (0, \infty)$ . The identity remains valid if we let  $1/x \rightarrow 1/x + i$ . This substitution has the effect of sending  $q \rightarrow -q$ . Combining the two identities and performing a great deal of simplification leads to

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}(1 - (-1)^{an+b})}{(xm)^2 + (an + b)^2} \\ &= -\frac{2\pi}{ax} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \left( \prod_{m=0}^{\infty} \frac{(1 - \omega^{2j+1}q^{2m+1})(1 - \omega^{-2j-1}q^{2m+1})}{(1 + \omega^{2j+1}q^{2m+1})(1 + \omega^{-2j-1}q^{2m+1})} \right). \end{aligned}$$

Taking note of (4.1), and then recalling the product representation for  $\operatorname{dn}(u)$  [1, p. 918], we find that the last expression transforms into

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}(1 - (-1)^{an+b})}{(xm)^2 + (an + b)^2} \\ &= \frac{2\pi}{ax} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \left( (1 - \alpha)^{1/4} \operatorname{dn} \left( \frac{(2j+1)K}{a} \right) \right). \end{aligned}$$

If we assume that  $a$  is even and  $b$  is odd, then the left-hand side of the identity equals  $2F_{(a,b)}(x)$ . We can recover (4.2) by noting that  $\sum_j \omega^{-(2j+1)b} = 0$  whenever  $(a, b) = 1$ .  $\square$

In order to provide an application of (4.2), we evaluate  $F_{(10,1)}(1)$  explicitly. While we restrict our attention to this single example, the method we describe extends to many additional values of  $F_{(a,b)}(x)$ . Let us recall that  $\operatorname{dn}(u)$  has real period  $2K$ . If we use the symmetries [13, p. 500]

$$\operatorname{dn}(2K - u) = \operatorname{dn}(u), \quad \operatorname{dn}(K - u) = \frac{\sqrt{1 - \alpha}}{\operatorname{dn}(u)}, \quad (4.3)$$

then (4.2) reduces to an expression involving  $[\frac{a-1}{4}]$  elliptic functions. When  $(a, b) = (10, 1)$ , we have

$$\begin{aligned} \frac{5x}{\pi} F_{(10,1)}(x) &= \sqrt{\frac{5 - \sqrt{5}}{2}} \log \left( \operatorname{dn} \left( \frac{K}{10} \right) \right) - \sqrt{\frac{5 + \sqrt{5}}{2}} \log \left( \operatorname{dn} \left( \frac{3K}{10} \right) \right) \\ &+ \frac{\sqrt{5 - 2\sqrt{5}}}{4} \log(1 - \alpha). \end{aligned} \quad (4.4)$$

By (4.1), it is possible to calculate  $\alpha$  whenever  $x^2 \in \mathbb{Q}^+$  and  $x > 0$ . It just remains to compute the values of the elliptic functions.

Notice that  $\operatorname{dn}(rK/s)$  is an algebraic function of  $\alpha$  if  $(r, s) \in \mathbb{Z}^2$ . This is a consequence of the fact that elliptic functions obey addition formulas [1, p. 574]. Perhaps the easiest method for calculating values such as  $\operatorname{dn}(K/10)$  and  $\operatorname{dn}(3K/10)$  is to generate polynomials (but not necessarily minimal ones) which they satisfy, by iterating the duplication formula for  $\operatorname{dn}(z)$  [1, p. 574]. Let us recall that

$$\operatorname{dn}(2z) = f(\operatorname{dn}(z)), \quad (4.5)$$

where

$$f(x) := \frac{x^2 + (x^2 - 1)(1 + \frac{1}{\alpha}(x^2 - 1))}{x^2 - (x^2 - 1)(1 + \frac{1}{\alpha}(x^2 - 1))}. \quad (4.6)$$

For brevity, we use the shorthand notation

$$d_j := \operatorname{dn} \left( \frac{jK}{10} \right).$$

Using (4.3) and (4.5), we can easily show that

$$\begin{aligned} d_{2j} &= f(d_j), \\ d_j &= d_{20-j}, \\ d_{10-j} &= \frac{\sqrt{1-\alpha}}{d_j}, \\ d_5 &= \sqrt[4]{1-\alpha}. \end{aligned}$$

As a consequence of the elementary properties above, it is easy to deduce that

$$\begin{aligned} f(d_1)f(f(f(d_1))) - \sqrt{1-\alpha} &= 0, \\ f(d_3)f(f(f(d_3))) - \sqrt{1-\alpha} &= 0. \end{aligned} \quad (4.7)$$

For instance, notice that  $f(d_1)f(f(f(d_1))) = d_2f(f(d_2)) = d_2f(d_4) = d_2d_8 = \sqrt{1-\alpha}$ . It follows immediately that  $\operatorname{dn}(K/10)$  and  $\operatorname{dn}(3K/10)$  are conjugate zeros of a function which is rational in  $\mathbb{Q}(\sqrt{1-\alpha})$ . It is easy to extract a polynomial which  $d_1$  and  $d_3$  satisfy, by considering only the numerator of  $f(x)f(f(f(x))) - \sqrt{1-\alpha}$ .

Now we can finish the computation of  $F_{(10,1)}(1)$ . Equation (4.1) shows that  $x = 1$  when  $\alpha = 1/2$ . As a result, (4.4) becomes

$$\begin{aligned} \frac{5}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{n+m}}{m^2 + (10n+1)^2} &= \sqrt{\frac{5-\sqrt{5}}{2}} \log(d_1) - \sqrt{\frac{5+\sqrt{5}}{2}} \log(d_3) \\ &\quad - \frac{\sqrt{5-2\sqrt{5}}}{4} \log 2, \end{aligned} \quad (4.8)$$

where

$$d_1 = \operatorname{dn}(K/10) \approx 0.9915\dots, \quad d_3 = \operatorname{dn}(3K/10) \approx 0.9309\dots$$

If we use `Mathematica` to expand (4.7), then it is easy to see that  $d_1$  and  $d_3$  are conjugate zeros of the irreducible polynomial

$$\begin{aligned} 0 = & 1 + 32X^2 - 1152X^4 + 14528X^6 - 103328X^8 + 445056X^{10} - 747008X^{12} \\ & - 5859584X^{14} + 67132864X^{16} - 404289024X^{18} + 1770485760X^{20} \\ & - 6097568768X^{22} + 17124502016X^{24} - 40180561920X^{26} + 80299532288X^{28} \\ & - 138787278848X^{30} + 209592829440X^{32} - 277574557696X^{34} \\ & + 321198129152X^{36} - 321444495360X^{38} + 273992032256X^{40} \\ & - 195122200576X^{42} + 113311088640X^{44} - 51748995072X^{46} \\ & + 17186013184X^{48} - 3000107008X^{50} - 764936192X^{52} \\ & + 911474688X^{54} - 423231488X^{56} + 119013376X^{58} - 18874368X^{60} \\ & + 1048576X^{62} + 65536X^{64}. \end{aligned}$$

The calculation is essentially complete, but we provide a few additional comments. Despite the fact that `Mathematica` could not solve this equation directly, it is possible to express  $d_1$  and  $d_3$  in terms of radicals, as we demonstrate below. It is unfortunate that the formulas are prohibitively complicated.

We conclude by briefly describing how to recover explicit formulas for  $d_1$  and  $d_3$ . First notice that if  $f(x) = y$ , with  $f(x)$  defined in (4.6), then we can express  $x$  in terms of  $y$  by solving quadratic equations. Since  $\operatorname{dn}(2K/5) = f(f(d_1))$  and  $\operatorname{dn}(6K/5) = f(f(d_3))$ , it is sufficient to reduce  $\operatorname{dn}(2K/5)$  and  $\operatorname{dn}(6K/5)$  to radicals. There are several methods to accomplish this calculation. The simplest approach is to generate their minimal polynomials by repeated applications of the duplication formula for  $\operatorname{dn}(z)$ . It is then possible to verify the formulas

$$\begin{aligned} \operatorname{dn}\left(\frac{2K}{5}\right) &= \frac{1}{4} \left( 1 + \sqrt{5} + 2\sqrt{2 + \sqrt{5}} - \sqrt{2(5 + \sqrt{5})} \right), \\ \operatorname{dn}\left(\frac{6K}{5}\right) &= \frac{1}{4} \left( 1 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}} + \sqrt{2(5 + \sqrt{5})} \right). \end{aligned}$$

In this example we have assumed that  $\alpha = 1/2$ . It is still possible, albeit significantly more difficult, to evaluate these elliptic functions for certain other values of  $\alpha$ .

## 5. CONCLUSION

We have shown how to prove many explicit formulas for  $F_{(a,b)}(x)$ . The most obvious extension of this research is to examine cases where  $a > 5$  is an odd integer. Notice that Theorem 4.1 does not apply to those values. It should also be interesting to attempt to apply our techniques to the class of sums studied by Zucker and McPhedran in [16]. They gave closed form evaluations for many values of

$$S(p, r, j) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{j^{2s}}{((jn + p)^2 + (jm + r)^2)^s},$$

in terms of Dirichlet  $L$ -series.

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